# Polynomial Eigenvalue Solutions to the 5-pt and 6-pt Relative Pose Problems

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**Abstract**

In this paper we provide new fast and simple solutions to two important minimal problems in computer vision, the five-point relative pose problem and the six-point focal length problem. We show that these two problems can easily be formulated as polynomial eigenvalue problems of degree three and two and solved using standard efficient numerical algorithms. Our solutions are somewhat more stable than state-of-the-art solutions by Nister and Stewenius and are in some sense more straightforward and easier to implement since polynomial eigenvalue problems are well studied with many efficient and robust algorithms available. The quality of the solvers is demonstrated in experiments [1].

## 1 Introduction

Estimating relative camera pose [11] from image correspondences can be formulated as a minimal problem and solved from a minimal number of image points [8]. It is known that for estimating relative pose of two fully-calibrated cameras, the minimal number of image points is five and for two cameras with unknown focal length it is six. Therefore, these two minimal problems are often called the *five-point relative pose problem* (5-pt) and the *six-point focal length problem* (6-pt).

Efficient algorithms for solving these two important minimal problems appeared only recently [19, 26, 23] and are widely used in many applications such as 3D reconstruction and structure from motion. It is because they are very effective as hypothesis generators in popular RANSAC paradigm [8] or can be used for initializing the bundle adjustment [11].

Using a small data set considerably reduces the number of samples in RANSAC. Therefore, new minimal problems [23, 25, 26, 10, 13, 14, 2, 4] have been solved recently. Many of them led to nontrivial systems of polynomial equations. A popular method for solving such systems is based on polynomial ideal theory and Gröbner bases [5]. The Gröbner basis method was used to solve almost all previously mentioned minimal problems including the 5-pt [26] and 6-pt problems [23]. This approach is general but not always straightforward and without a special care [3] it may lead to procedures that are not numerically robust [24, 14]. Automatic generator of such Gröbner basis solvers has been presented in [15].

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In this paper we show that the 5-pt and the 6-pt problems lead to special polynomial equations that can be solved robustly and efficiently as a cubic and a quadratic eigenvalue problems. This solution is fast and somewhat more stable than existing solutions [19, 26, 23]. It is in some sense also more straightforward and easier to implement since polynomial eigenvalue problems are well studied and their efficient numerical solvers are available. Polynomial eigenvalue solvers were previously used to solve the problem of autocalibration of one-parameter radial distortion from nine point correspondences [9] or to estimate paracatadioptric camera model from image matches [18]. Motivated by all these examples, we also characterize problems that can be solved by polynomial eigenvalue solvers.

# 2 The five and six point relative pose problems

Consider a pair of pin-hole cameras $\mathtt{P}$ and $\mathtt{P}'$. The constraints on corresponding image points in two calibrated views can be written down as [11]:

$$\mathbf{x}_j'^T \mathtt{E} \mathbf{x}_j = 0, \tag{1}$$

where $\mathtt{E}$ is a $3 \times 3$ rank-2 essential matrix, and it is known that

$$\det(\mathtt{E}) = 0, \tag{2}$$

$$2\mathtt{E}\mathtt{E}^\top\mathtt{E} - trace(\mathtt{E}\mathtt{E}^\top)\mathtt{E} = 0. \tag{3}$$

The usual way to compute essential matrix is to linearize relation (1) into form $\mathtt{M}X = 0$, where vector $X$ contains nine elements of the matrix $\mathtt{E}$ and $\mathtt{M}$ contains image measurements. Essential matrix $\mathtt{E}$ is then constructed as a linear combination of the null space vectors of the matrix $\mathtt{M}$. The dimension of the null space depends on the number of point correspondences used. Additional constraints (2) and (3) are used to determine the coefficients in the linear combination of the null space vectors or to project an approximate solution to the space of correct essential matrices.

In this paper we also consider a camera pair with unknown but constant focal lengths $f$. Other calibration parameters are known. Calibration matrix $\mathtt{K}$ is then a diagonal matrix $diag([f\ f\ 1])$. It is known that for such configuration $\mathtt{E} = \mathtt{K}\mathtt{F}\mathtt{K}$. Since $\mathtt{K}$ is regular we have

$$\det(\mathtt{F}) = 0, \tag{4}$$

$$2\mathtt{F}\mathtt{Q}\mathtt{F}^\top\mathtt{Q}\mathtt{F} - trace(\mathtt{F}\mathtt{Q}\mathtt{F}^\top\mathtt{Q})\mathtt{F} = 0. \tag{5}$$

The equation (5) is obtained by substituting the expression for the essential matrix into the trace constraint (3), applying the substitution $\mathtt{Q} = \mathtt{K}\mathtt{K}$, and multiplying (3) by $\mathtt{K}^{-1}$ from left and right. Note that the calibration matrix can be written as $\mathtt{K} \simeq diag([1\ 1\ 1/f])$, what simplifies equations.

# 3 Previous solutions

## 3.1 Five point problem

The 5-pt relative pose problem was studied already by Kruppa [12] who has shown that it has at most eleven solutions. Maybank and Faugeras [7] then sharpened Kruppa's result by showing that there are at most ten solutions. Recently, Nister et al. [20] have shown that the problem really requires solving a ten degree polynomial.

Kruppa [12] also gave an algebraic algorithm for solving the 5-pt problem. It was implemented by Maybank and Faugearas [7] but turned out not to be particularly efficient and practical. More efficient and practical solution has been invented by Philip [21]. He designed a method finding the solutions by extracting the roots of a thirteen degree polynomial. The state of the art methods of Nister [19] and Stewénius et al. [26], which obtain the solutions as the roots of a tenth-degree polynomial, are currently the most efficient and robust implementations for solving the 5-pt relative pose problem.

In both these methods, the five linear epipolar constraints were used to parametrize the essential matrix as a linear combination of a basis $\mathtt{E}_1, \mathtt{E}_2, \mathtt{E}_3, \mathtt{E}_4$ of the space of all compatible essential matrices

$$E = x\mathtt{E}_1 + y\mathtt{E}_2 + z\mathtt{E}_3 + \mathtt{E}_4. \tag{6}$$

Then, the rank constraint (2) and the trace constraint (3) were used to build ten third-order polynomial equations in three unknowns and 20 monomials. These equations can be written in a matrix form

$$\mathtt{M}X = 0, \tag{7}$$

with a coefficient matrix $\mathtt{M}$ reduced by the Gauss-Jordan (G-J) elimination and the vector of all monomials $X$.

The method [19] used relations between polynomials (7) to create three additional equations. The new equations were arranged into a $3 \times 3$ matrix equation $\mathtt{A}(z)Z = 0$ with matrix $\mathtt{A}(z)$ containing polynomial coefficients in $z$ and $Z$ containing the monomials in $x$ and $y$. The solutions were obtained by solving the tenth degree polynomial $\det(\mathtt{A}(\mathtt{z}))$, finding $Z$ as a solution to a homogeneous linear system, and constructing $\mathtt{E}$ from (6).

The method [26] follows a classical approach to solving systems of polynomial equations. First, a Gröbner basis [5] of the ideal generated by equations (7) is found. Then, a *multiplication matrix* [22] is constructed. Finally, the solutions are obtained by computing the eigenvectors [5] of the multiplication matrix. This approach turned out to lead to a particularly simple procedure for the 5-pt problem since a Gröbner basis and a $10 \times 10$ multiplication matrix can be constructed directly from the reduced coefficient matrix $\mathtt{M}$.

Another technique, based on the hidden variable resultant, for solving the 5-pt relative pose problem was proposed in [17]. This technique is somewhat easier to understand than [26] but is far less efficient and Maple was used to evaluate large determinants.

## 3.2 Six point problem

The problem of estimating relative camera position for two cameras with unknown focal length from minimal number of point correspondences has 15 solutions.

The first minimal solution to this problem proposed by Stewénius et. al. [23] is based on the Gröbner basis techniques and is similar to the Stewénius' solution to the 5-pt problem [26]. Using the linear epipolar constraints, the fundamental matrix is parameterized by two unknowns as

$$F = xF_1 + yF_2 + F_3. \tag{8}$$

Using the rank constraint for the fundamental matrix (4) and the trace constraint for the essential matrix (5) then brings ten third and fifth order polynomial equations in three unknowns $x$, $y$, and $w = f^{-2}$, where $f$ is the unknown focal length.

The Gröbner basis solver [23] starts with these ten polynomial equations which can be represented by a $10 \times 33$ matrix $\mathtt{M}$. Since this matrix doesn't contain all necessary polynomials for creating a multiplication matrix, two new polynomials are added and matrix $\mathtt{M}$ is reduced by the G-J elimination. Further four new polynomials are added and eliminated then. Finally, two more polynomials are added and eliminated. The resulting system then contains a Gröbner basis and can be used to construct the multiplication matrix. The resulting solver therefore consists of three G-J eliminations of three matrices of size $12 \times 33$, $16 \times 33$, and $18 \times 33$. The eigenvectors of the multiplication matrix provide the solutions to the three unknowns $x$, $y$, and $w = f^{-2}$.

Another Gröbner basis solver to this problem was proposed in [3]. This solver uses only one G-J elimination of a $34 \times 50$ matrix and uses a special technique for improving the numerical stability of Gröbner basis solvers based on changing the basis $B$. In this paper it was shown that this solver gives more accurate results than the original solver [23].

A Gröbner basis solver with single G-J elimination of a $31 \times 46$ matrix which was generated using automatic generator has been presented in [15].

Solution based on the hidden variable resultant method was proposed in [16]. This solution has similar problems as the hidden variable solution to the five point problem [17].

# 4 Polynomial eigenvalue problems

Polynomial eigenvalue problems (PEP) are problems of the form

$$\mathtt{A}(\lambda)\,\mathbf{v} = 0, \tag{9}$$

where $\mathtt{A}(\lambda)$ is a matrix polynomial defined as

$$\mathtt{A}(\lambda) \equiv \lambda^l \mathtt{C}_l + \lambda^{l-1}\mathtt{C}_{l-1} + \cdots + \lambda \mathtt{C}_1 + \mathtt{C}_0, \tag{10}$$

in which the $\mathtt{C}_j$ are square $n$ by $n$ matrices.

To illustrate what kind of systems of polynomial equations can be transformed to PEP consider a system of equations

$$f_1(\mathbf{x}) = ... = f_m(\mathbf{x}) = 0, \tag{11}$$

which are given by a set of $m$ polynomials $\mathtt{F} = \{f_1, ..., f_m | f_i \in \mathbb{C}[x_1, ..., x_n]\}$ in $n$ variables $\mathbf{x} = (x_1, ..., x_n)$ over the field of complex numbers. Each of these equations can be written as $f_i = \sum_{j=1}^{k} a_{ij}\mathbf{x}^{\alpha(j)}$, $i = 1, \ldots, m$, where $\mathbf{x}^{\alpha(j)} = x_1^{\alpha(j)_1} x_2^{\alpha(j)_2} ... x_n^{\alpha(j)_n}$, $j = 1, \ldots, k$ are $k$ monomials which appear in these equations and $a_{ij}$ are coefficient of these equations. If for some $x_j$ these equations can be rewritten as

$$f_i = \sum_{p=0}^{l} x_j^p \sum_{q=1}^{r} b_{p_{iq}} \overline{\mathbf{x}}^{\beta(q)}, i = 1, \ldots, m, \tag{12}$$

where $b_{p_{iq}}$ are coefficients and $\overline{\mathbf{x}}^{\beta(q)}$ are monomials in all variables except $x_j$, $\overline{\mathbf{x}}^{\beta(q)} = x_1^{\beta(q)_1} \ldots x_{j-1}^{\beta(q)_{j-1}} x_{j+1}^{\beta(q)_{j+1}} \ldots x_n^{\beta(q)_n}$ and if $r = m$, i.e. that we have so many equations as monomials in these equations (monomials $\overline{\mathbf{x}}^{\beta(q)}$ without the variable $x_j$), then the system of polynomial equations (11) can be transformed to a PEP (9) of degree $l$.

An important class of polynomial eigenvalue problems are quadratic eigenvalue problems where the matrix polynomial $\mathtt{A}(\lambda)$ has degree two.

Quadratic eigenvalue problem (QEP) is a problem of the form

$$\left(\lambda^2 \mathtt{C}_2 + \lambda \mathtt{C}_1 + \mathtt{C}_0\right) \mathbf{v} = 0, \tag{13}$$

where $\mathtt{C}_2, \mathtt{C}_1$ and $\mathtt{C}_0$ are given matrices of size $n \times n$ and $x$ is the the eigenvector and $\lambda$ corresponding eigenvalue.

QEP (13) can be easily transformed to the following generalized "linear" eigenvalue problem (GEP) [1]

$$\mathtt{A}\mathbf{y} = \lambda \mathtt{B}\mathbf{y}, \tag{14}$$

where

$$A = \begin{pmatrix} 0 & I \\ -\mathtt{C}_0 & -\mathtt{C}_1 \end{pmatrix}, \mathtt{B} = \begin{pmatrix} \mathtt{I} & 0 \\ 0 & \mathtt{C}_2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} \mathbf{v} \\ \lambda \mathbf{v} \end{pmatrix}. \tag{15}$$

It can be easily seen that QEP (13) has $2n$ eigenvalues.

Generalized eigenvalue problems (14) are well studied problems and there exist many efficient numerical algorithms for solving them [1]. For example, MATLAB provides the function polyeig for solving directly polynomial eigenvalue problems (9) of arbitrary degree (including their transformation to the generalized eigenvalue problems).

Some applications lead to a higher order polynomial eigenvalue problems (PEP)

$$\left(\lambda^l \mathtt{C}_l + \lambda^{l-1} \mathtt{C}_{l-1} + \cdots + \lambda \mathtt{C}_1 + \mathtt{C}_0\right) \mathbf{v} = 0, \tag{16}$$

in which the $\mathtt{C}_j$ are square $n \times n$ matrices.

As for the QEP, these problems can be transformed to the generalized eigenvalue problem (14). Here

$$\mathtt{A} = \begin{pmatrix} 0 & \mathtt{I} & 0 & \dots & 0 \\ 0 & 0 & \mathtt{I} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\mathtt{C}_0 & -\mathtt{C}_1 & -\mathtt{C}_2 & \dots & -\mathtt{C}_{l-1} \end{pmatrix}, \mathtt{B} = \begin{pmatrix} \mathtt{I} & & & \\ & \dots & & \\ & & \mathtt{I} & \\ & & & \mathtt{C}_l \end{pmatrix}, \mathbf{y} = \begin{pmatrix} \mathbf{x} \\ \lambda \mathbf{x} \\ \dots \\ \lambda^{l-1} \mathbf{x} \end{pmatrix}. \tag{17}$$

For higher order PEPs, one has to work with larger matrices with $n\,l$ eigenvalues. Therefore, for larger values of $n$ and $l$ problems with the convergence of the techniques for solving these problems sometimes appear. One way how to solve these convergence problems is to project the given problem (16) onto a low-dimensional subspace and obtain a similar problem of low dimension [1]. This low-dimensional PEP can then be easier solved using standard methods.

# 5 Polynomial eigenvalue formulation of 5-pt & 6-pt

In this section we solve the 5-pt relative pose problem and the 6-pt focal length problem as the polynomial eigenvalue problems (9) of degree three and degree two.

## 5.1 Five point problem

In our solution we use the same formulation of the five point problem as it was used in [19] and [26]. We first use linear equations from the epipolar constraint to parametrize the essential matrix with three unknowns $x$, $y$, and $z$ (6). Using this parameterization, the rank (2) and the trace constraints (3) results in ten third-order polynomial equations in three unknowns and 20 monomials and can be written in a matrix form

$$\mathtt{M}X = 0, \tag{18}$$

where $\mathtt{M}$ is a $10 \times 20$ coefficient matrix and $X = (x^3, yx^2, y^2x, y^3, zx^2, zyx, zy^2, z^2x, z^2y, z^3, x^2, yx, y^2, zx, zy, z^2, x, y, z, 1)^\top$ is the vector of all monomials. Between these monomials there are all monomials in all three unknowns up to degree three. So we can use arbitrary unknown to play the role of $\lambda$ in (9). For example, taking $\lambda = z$, these ten equations can be easily rewrite as

$$\left(z^3\mathtt{C}_3 + z^2\mathtt{C}_2 + z\,\mathtt{C}_1 + \mathtt{C}_0\right)\mathbf{v} = 0, \tag{19}$$

where $\mathbf{v}$ is a $10 \times 1$ vector of monomials, $\mathbf{v} = (x^3, x^2y, xy^2, y^3, x^2, xy, y^2, x, y, 1)^\top$ and $\mathtt{C}_3$, $\mathtt{C}_2$, $\mathtt{C}_1$ and $\mathtt{C}_0$ are $10 \times 10$ coefficient matrices. $\mathtt{C}_3 \equiv (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ m_{10})$, $\mathtt{C}_2 \equiv (0\,0\,0\,0\,0\,0\,0\,m_8\ \ m_9\ m_{16})$, $\mathtt{C}_1 \equiv (0\,0\,0\,0\,m_5\ m_6\ m_7\ m_{14}\ m_{15}\ m_{19})$, and $\mathtt{C}_0 \equiv (m_1\ m_2\ m_3\ m_4\ m_{11}\ m_{12}\ m_{13}\ m_{17}\ m_{18}\ m_{20})$, where $m_j$ is the $j^{th}$ column from the coefficient matrix $\mathtt{M}$.

Since $\mathtt{C}_3$, $\mathtt{C}_2$, $\mathtt{C}_1$ and $\mathtt{C}_0$ are known square matrices, the formulation (19) is a cubic PEP. Such problem can be easily solved using standard efficient algorithms, for example MATLAB function `polyeig`.

After solving the PEP (19), we obtain 30 eigenvalues $z$ and 30 corresponding eigenvectors $\mathbf{v}$ from which we extract solutions for $x$ and $y$. Then we use (6) to compute $\mathtt{E}$.

## 5.2 Six point problem

Our solution to the 6-pt focal length problem starts with the parameterization of the fundamental matrix with two unknowns $x$ and $y$ (8). Substituting this parameterization into the rank constraint for the fundamental matrix (4) and the trace constraint for the essential matrix (5) gives ten third and fifth order polynomial equations in the three unknowns $x$, $y$ and $w = f^{-2}$, where $f$ is the unknown focal length. This formulation is the same as the one used in [23, 16] and can be again written in a matrix form

$$\mathtt{M}X = 0, \tag{20}$$

where $\mathtt{M}$ is a $10 \times 30$ coefficient matrix and $X = (w^2x^3, w^2yx^2, w^2y^2x, w^2y^3, wx^3, wyx^2, wy^2x, wy^3, w^2x^2, w^2yx, w^2y^2, x^3, yx^2, y^2x, y^3, wx^2, wyx, wy^2, w^2x, w^2y, x^2, yx, y^2, wx, wy, w^2, x, y, w, 1)^\top$ is a vector of 30 monomials. In these $x$ and $y$ appear in degree three and $w$ only in degree two. Therefore, we have selected $\lambda = w$. Then, these ten equations can be easily rewrite as

$$\left(w^2\mathtt{C}_2 + w\,\mathtt{C}_1 + \mathtt{C}_0\right)\mathbf{v} = 0, \tag{21}$$

where $\mathbf{v}$ is a $10 \times 1$ vector of monomials, $\mathbf{v} = (x^3, x^2y, xy^2, y^3, x^2, xy, y^2, x, y, 1)^\top$ and $\mathtt{C}_2, \mathtt{C}_1$ and $\mathtt{C}_0$ are $10 \times 10$ coefficient matrices. $\mathtt{C}_2 \equiv (m_1\ m_2\ m_3\ m_4\ m_9\ m_{10}\ m_{11}\ m_{19}\ m_{20}\ m_{26})$, $C_1 \equiv (m_5\ m_6\ m_7\ m_8\ m_{16}\ m_{17}\ m_{18}\ m_{24}\ m_{25}\ m_{29})$ and $\mathtt{C}_0 \equiv (m_{12}\ m_{13}\ m_{14}\ m_{15}\ m_{21}\ m_{22}\ m_{23}\ m_{27}\ m_{28}\ m_{30})$ , where $m_j$ is the $j^{th}$ column from the corresponding coefficient matrix $\mathtt{M}$.

The formulation (21) is a QEP. We can again easily solve this problem using standard efficient algorithms.

After solving the QEP (21) we obtain 20 eigenvalues $w$ and 20 corresponding eigenvectors $\mathbf{v}$ from which we extract solutions for $x$ and $y$ and use (8) to get solutions for focal length $f$ and fundamental matrix $\mathtt{F}$.

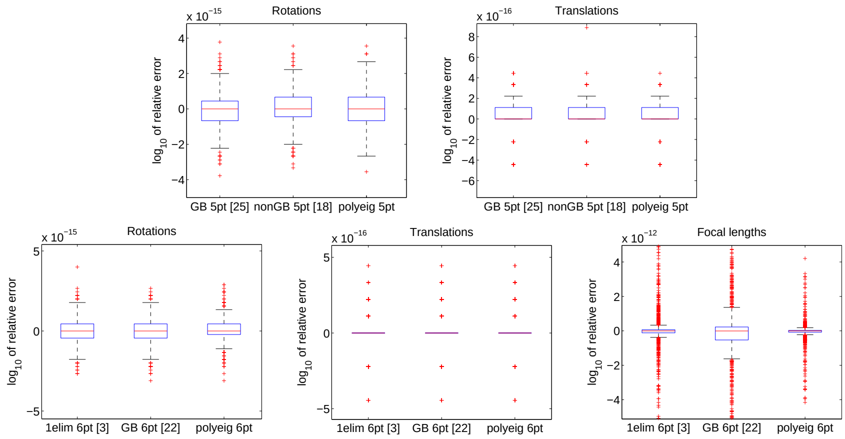


Figure 1: Noise free test. The results are presented by the MATLAB function boxplot which shows values 25% to 75% quantile as a blue box with red horizontal line at median. The red crosses show data beyond 1.5 times the interquartile range.

# 6 Experiments

In this section we evaluate both our solutions and compare them with the existing state of the art methods. Since all methods are algebraically equivalent and solvers differ only in the way of solving problems, we have evaluated them on synthetic data sets only. We aimed at studying numerical stability and speed of the algorithms in several configurations.

In all our tests, the synthetic scene was generated randomly with the Gaussian distributions in a 3D cube. Each 3D point was projected by a pair of cameras, where each camera orientation and position were selected depending on the testing configuration. Then, Gaussian noise with standard deviation $\sigma$ was added to each image point.

For the calibrated 5-pt problem we extracted camera relative rotations and translations from estimated essential matrices. From the four possible choices of rotations and translations we selected the one where all 3D points were in the front of the canonical camera pair [11]. Let $\mathtt{R}$ be an estimated camera relative rotation and $\mathtt{R}_{gt}$ a ground-truth rotation. The rotation error is measured as an angle in the angle axis representation of the relative rotation $\mathtt{R}\mathtt{R}_{gt}^{-1}$ and the translation error as an angle between ground-truth and estimated translation vectors.

In the 6-pt problem evaluations we used estimated focal length to transfer fundamental matrix to an essential matrix and then we measured errors as in the 5pt problem case. We also measured relative focal length error $(f - f_{gt})/f_{gt}$, where $f$ is an estimated focal length and $f_{gt}$ denotes a ground truth value.

In the first test we studied numerical stability of our algorithms in noise free situation. Here we show rotation and translation errors w.r.t. the ground truth as a *cosine* of the angles which we get as described above. This is because the errors are so small that the rounding in *acos* function removed differences between the algorithms. Hence we skipped *acos* in the calculations. Figure 1 summarises results for 5000 randomly generated scenes and camera configurations. From the figure we see that all algorithms behaved

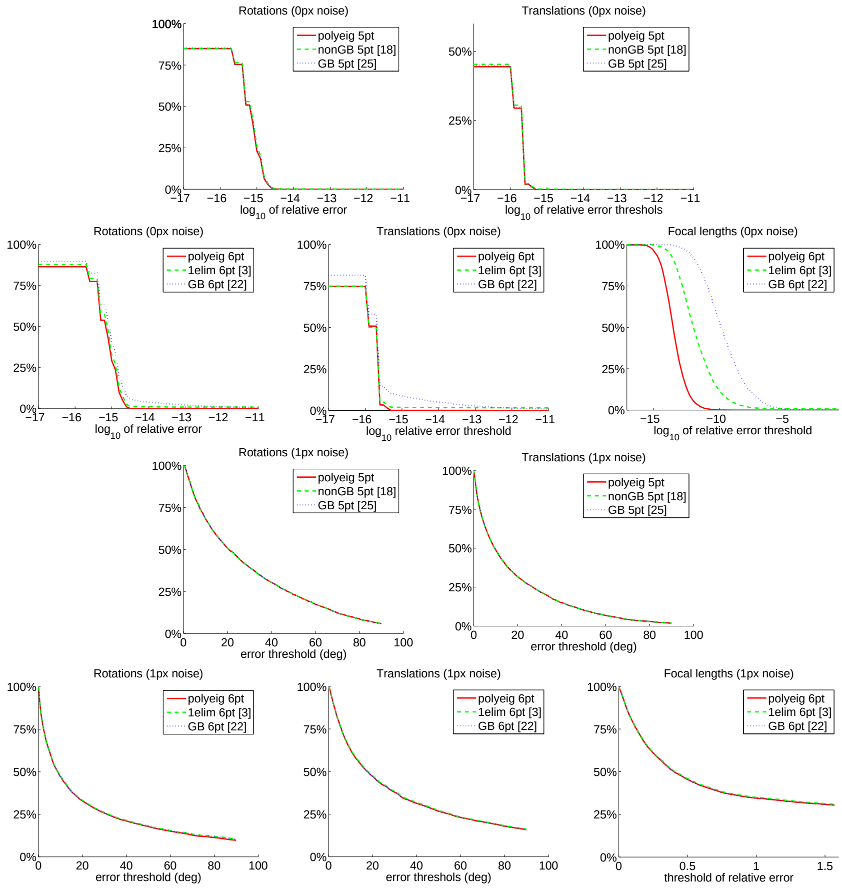


Figure 2: Top five plots show failure ratio test for noise free data. Tests with data affected by a 1 pxl noise ($1000 \times 1000$ image resolution) are displayed in the bottom.

almost the same way. However, 6pt polyeig solution provided more precise focal length estimation than existing 3 elimination [23] as well as 1-elimination Gröbner basis method [15]. deviation $\sigma$ did not cause observable change in relative behaviour of the algorithms, therefore we do not plot these results.

Next, we studied the fraction of results (y-axis) that gives worse result than a given error threshold (x-axis). We generated 5000 random scenes and camera poses. Figure 2 summarizes these results for both noise free data (top two rows) and for data affected by a 1 *pxl* noise (bottom two rows). Our polyeig solvers provided more stable results than the state of the art methods.

In the last experiment we tested behaviour of the algorithms for the camera pair with a constant baseline length (0.5m) moving backwards and zooming to keep image filled by the projection of the 3D scene. For each position of the cameras pair, we executed all algorithms several times and selected median values from measured errors. Results are displayed in Figure 3. From the plots we see that the original 6-pt algorithm returns less precise results with increasing distance and focal length. Both, single elimination

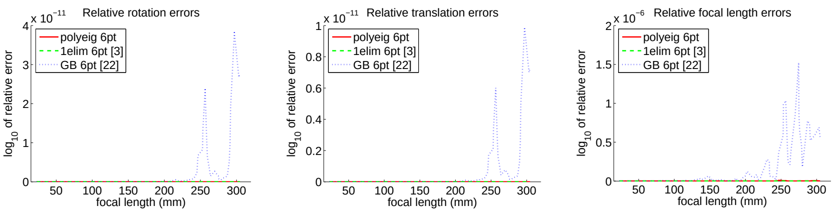


Figure 3: Backward moving and zooming camera test.

Gröbner basis method and our polynomial eigenvalue method, return comparable results.

We have implemented our solvers fully in MATLAB. Solvers are easy to implement since only three (for the 6-pt problem) and four (5-pt problem) matrices are filled with appropriate coefficients extracted from the input equations followed by MATLAB *polyeig* function and constructing `E` and `F` from equations (6) and (8).

Comparing the speed of the algorithms, our MATLAB implementation of the 6-pt problem solver is $3\times$ slower (3 ms) than the MEX implementation of the original solution (1 ms) proposed by Stewenius [23]. Our 6-pt solver is about 30% faster than the MATLAB implementation of the single elimination Gröbner basis 6-pt solver with a MEX implementation of the Gauss-Jordan elimination. Our *polyeig* solution of the 5-pt problem is about $8\times$ slower (8 ms) than the Gröbner basis MEX solution (1 ms). On the other hand, according to the MATLAB profiler tool, eigenvalue computation was not the most expensive part of our solvers. Hence we believe, that the MEX version of our solvers can be made faster.

# 7 Conclusion

In this paper we have presented a new simple and numerically stable solutions to the two important minimal problems in computer vision, the five-point relative pose problem and the six-point focal length problem. We have formulated them as a quadratic and cubic polynomial eigenvalue problems which can be solved robustly and efficiently using existing numerical methods. Experiments have shown that our solutions are somewhat more stable than the state-of-the-art methods [19, 26, 23]. Our solutions are easy to implement while achieving comparable speed and numerical accuracy.

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