



Estimating the Shape of A Surface with Non-Constant Reflectance from a Single Color Image

Ambrose E. Ononye and Philip W. Smith
Electrical and Computer Engineering
The University of Tennessee
Knoxville, TN, 37996, USA
philsmith@utk.edu

Abstract

Most shape estimation algorithms assume images of smooth surfaces with constant surface reflectance and often require knowledge of illuminant positions. These two requirements limit the applicability of such approaches to a small subset of imaged objects. In this paper a variational approach to the shape-from-color problem is presented that requires no knowledge of scene illuminants, and makes no assumptions regarding an object's reflectance properties. Using a surface smoothness constraint, this technique can accurately reconstruct the shape of any continuous Lambertian surface given only a single color image. The capabilities of this algorithm are demonstrated using synthetic and real imagery.

1 Introduction

Humans have the ability to infer both the surface reflectance and three-dimensional shape of objects from two-dimensional photographs under simple and complex illumination fields. A central theme of research in computer vision has focused on understanding and mimicking this ability of humans to reconstruct object shape from two-dimensional visual cues due to the central role shape plays in a variety of applications. Unfortunately, most of the reported algorithms in the area rely on a number of simplifying assumptions that degrade their performance in uncontrolled environments.

Shape estimation is defined as the recovery of a discrete set of surface normals which represent the orientation of local areas along a continuous surface. Due to the physics of light reflection at material surfaces [1], the apparent image intensity of a surface patch at a given wavelength is a function of the angle between the surface normal and the direction cosines of the incident illuminants. Hence, both gray-scale and color image intensity information can be used to estimate surface shape from a single image.

Local surface orientation has three degrees of freedom, but gray-scale intensity information and normal conditions provide only two analytical constraints. The under-constrained nature of this problem results in an ambiguity in the recovered shape. Thus, the primary challenge in the development of shape-from-shading algorithms using gray-scale imagery lies in uncovering other sensory or heuristic information which can be



used to further constrain the process. Various additional criteria have been proposed and employed for shape estimation, the most popular being smoothness [2, 3, 4], or more recently, symmetry [5]. While techniques based on these constraints perform well in controlled conditions, they assume surfaces with constant reflectance, or apparent color, and often require knowledge of illuminant positions, severely restricting their application to images of typical objects acquired outside the laboratory.

As reported by Drew and Kontsevich [6, 7], a closed-form linear solution exists for estimating both surface orientation and illumination parameters up to an orthogonal transformation given a single RGB color image. Using the orientation normality constraint, the illumination function is first estimated and then directly employed in the computation of surface normals. While this shape-from-color method is able to recover shapes given images acquired in uncontrolled lighting environments, the apparent color of the surface must remain constant for accurate estimation.

For a single-image shape estimation algorithm to be applicable in general environments, it must not require either 1) knowledge of illumination conditions, or 2) surfaces of constant reflectance. In this paper, we present a new technique for the automatic recovery of surface shape using color images that meets the above two criteria. Through the explicit use of a variational smoothness criteria, this iterative approach compensates for the orientation errors which arise from applying the existing shape-from-color technique to surfaces with non-constant reflectance.

2 Problem Overview

For color images, the apparent image intensity of points on a three-dimensional Lambertian surface, $\vec{I}(x, y) \in \mathcal{R}^3$, can be modeled using the expression

$$\vec{I}(x, y) = \mathbf{S}(x, y)\mathbf{L}(x, y)\mathbf{N}(x, y)\vec{n}(x, y) \quad (1)$$

where $\mathbf{S}(x, y) \in \mathcal{R}^{3 \times 3}$ is a diagonal matrix representing the surface reflectance function, $\mathbf{L}(x, y) \in \mathcal{R}^{3 \times 3}$ is the illuminant spectra function, $\mathbf{N}(x, y) \in \mathcal{R}^{3 \times 3}$ is the illuminant position function, and $\vec{n}(x, y) \in \mathcal{R}^3$ is the surface normal map of the imaged object [1]. Letting

$$\mathbf{M}(x, y) = \mathbf{S}(x, y)\mathbf{L}(x, y)\mathbf{N}(x, y),$$

equation 1 can be rewritten in terms of apparent color as

$$\vec{n}(x, y) = \mathbf{M}(x, y)^{-1}\vec{I}(x, y). \quad (2)$$

This equation represents the fundamental shape-from-color mapping — a linear transformation which relates the apparent color to orientation for each surface point. Under controlled conditions where the light source positions, light source spectra, and surface reflectance function of the object are known, shape can be recovered from the apparent color of pixels in a single image via a simple matrix inversion. Under normal imaging conditions, however, the illumination and surface reflectance functions are typically unknown.

If the imaged surface is not textured or self-occluding, however, the shape-from-color mapping, \mathbf{M} , is constant over (x, y) , and can be recovered up to an orthogonal transformation [6]. By definition, the inner product of the surface orientation at a point on the



surface must be such that

$$\vec{n}(x, y)^T \vec{n}(x, y) = 1. \quad (3)$$

Through substitution of equation 2, this normality condition provides a quadratic constraint on the entries of the shape-from-color mapping,

$$\begin{aligned} (\mathbf{M}^{-1} \vec{I}(x, y))^T (\mathbf{M}^{-1} \vec{I}(x, y)) &= 1 \\ \vec{I}^T(x, y) \mathbf{M}^{-T} \mathbf{M}^{-1} \vec{I}(x, y) &= 1 \\ \vec{I}^T(x, y) \mathbf{\Gamma} \vec{I}(x, y) &= 1. \end{aligned} \quad (4)$$

Hence, every pixel of the imaged surface must fall on the color-space ellipse given by

$$\begin{aligned} \gamma_{11} I_{1,n}^2 + \gamma_{22} I_{2,n}^2 + \gamma_{33} I_{3,n}^2 \\ + 2\gamma_{12} I_{1,n} I_{2,n} + 2\gamma_{13} I_{1,n} I_{3,n} + 2\gamma_{23} I_{2,n} I_{3,n} = 1 \end{aligned} \quad (5)$$

where γ_{ij} are the entries of the matrix $\mathbf{\Gamma}$, and $I_{k,n}$ are the k entries of the pixel intensity vector, $\vec{I}(x_n, y_n)$. Given N surface pixels in the image, there are N such equations which can be used to form the linear system

$$\mathbf{F} \vec{z} = \mathbf{1}, \quad (6)$$

where $\vec{z} \in \mathcal{R}^6$ such that

$$\vec{z} = [\gamma_{11} \ \gamma_{22} \ \gamma_{33} \ \gamma_{12} \ \gamma_{13} \ \gamma_{23}]^T$$

and $\mathbf{F} \in \mathcal{R}^{N \times 6}$ such that

$$\mathbf{F} = \begin{bmatrix} I_{1,1}^2 & I_{2,1}^2 & I_{3,1}^2 & I_{1,1} I_{2,1} & I_{1,1} I_{3,1} & I_{2,1} I_{3,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I_{1,N}^2 & I_{2,N}^2 & I_{3,N}^2 & I_{1,N} I_{2,N} & I_{1,N} I_{3,N} & I_{2,N} I_{3,N} \end{bmatrix}.$$

Solving equation 6 for \vec{z} produces an estimate of $\mathbf{\Gamma}$. The shape-from-color matrix, \mathbf{M}_e , can then be computed up to a rotation using Cholesky decomposition [8] and employed to construct an estimate of the surface orientation map, $\vec{n}_e(x, y)$, using equation 2.

The quadratic nature of the mapping constraint equation 4 implies that the estimated shape-from-color matrix, \mathbf{M}_e , is related to the actual mapping, \mathbf{M} , by an orthogonal transformation, $\mathbf{R} \in \mathcal{R}^{3 \times 3}$, such that

$$\mathbf{M}_e = \mathbf{M} \mathbf{R}.$$

As a result, the surface normal function recovered using \mathbf{M}_e and equation 2, is also related to the actual orientation map by the inverse rotation,

$$\vec{n}_e(x, y) = \mathbf{M}_e^{-1} \vec{I}(x, y) = (\mathbf{M} \mathbf{R})^{-1} \vec{I}(x, y) = \mathbf{R}^{-1} \mathbf{M}^{-1} \vec{I}(x, y) = \mathbf{R}^{-1} \vec{n}(x, y) \quad (7)$$

This result is not surprising due to the bi-linear nature of the lighting model. Various methods have been proposed for estimating this unknown rotation based on the heuristic assumption that the surface of interest is facing the camera [6]. Recovery of this rotation is often ignored when knowledge of the surface pose is not necessary for a given application, and is not addressed in this paper.

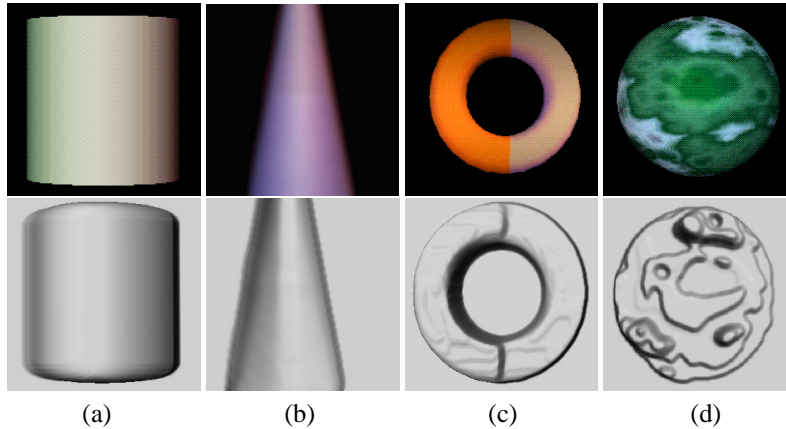


Figure 1: Surface shapes computed using the closed-form shape-from-color method described in section 2. Note that changes in surface color result in erroneous shape estimates.

Figures 1(a) and (b) show two different surfaces recovered using the shape recovery process for unknown illuminants described above. The shape-from-color algorithm is able to accurately recover shape for these objects where both the constant color and non-self-occluding assumptions are satisfied. When the constant surface reflectance assumption is not satisfied, however, the previously reported technique is unable to accurately reconstruct surface shapes as demonstrated in figures 1(c) and (d).

The inability of the existing shape-from-color technique to perform properly in the presence of color variations is a direct consequence of the ellipsoid-based method employed to recover the illumination matrix. When the apparent color of the surface is constant, the intensity vector of each surface pixel will fall on a single ellipse whose coefficients can be computed using the linear system of 6. The apparent intensity of surface pixels will fall on K different color-space ellipsoids, however, given a surface with K colors. If the linear system 6 is then used to recover the shape-from-color transform, the resulting parameters will belong to a single ellipse which is the ‘best-fit’ to all K ellipsoids in a least square sense. Ignoring the unknown global rotation, the relation between the estimated orientation, $\vec{n}_e(x, y)$, and actual surface normal, $\vec{n}(x, y)$, for an arbitrary surface point will be given by

$$\begin{aligned}
 \vec{n}_e(x, y) &= \mathbf{M}_e^{-1} \vec{I}(x, y) \\
 &= \mathbf{M}_e^{-1} (\tilde{\mathbf{S}}(x, y) \mathbf{M}_e \vec{n}(x, y)) \\
 &= (\mathbf{M}_e^{-1} \tilde{\mathbf{S}}(x, y) \mathbf{M}_e) \vec{n}(x, y) \\
 &= \mathbf{E}(x, y) \vec{n}(x, y)
 \end{aligned} \tag{8}$$

where $\tilde{\mathbf{S}}(x, y) = \mathbf{S}(x, y) \mathbf{S}_c^{-1}$ is the point-wise adjusted surface reflectance, and \mathbf{S}_c is the ‘constant’ surface reflectance that corresponds to the best-fit ellipsoid. This undesired transformation expresses itself as erroneous surface discontinuities at surface color boundaries as clearly demonstrated in figure 1(c).



3 A Variational Approach to the Shape-from-Color Problem

The existing closed-form solution to the simultaneous shape-from-color transform and orientation estimation problem discussed in the previous section is unable to accurately estimate shape when the reflectance of the imaged surface is not constant because it does not strictly enforce surface smoothness. A new variational algorithm is presented in this section that is able to estimate surface shape in the presence of reflectance variations through the explicit use of a continuity constraint.

Given initial estimates of both the shape-from-color mapping, $\mathbf{M}(x, y)$, and the surface orientation map, $\vec{n}(x, y)$, the total apparent image intensity error associated with those estimates is expressed as

$$\epsilon = \sum_{x,y} \|\vec{I}(x, y) - \mathbf{M}(x, y)\vec{n}(x, y)\|^2$$

Thus, possible solutions to the shape-from-color problem satisfy the expression

$$\min_{\mathbf{M}, \vec{n}} \epsilon = \sum_{x,y} \|\vec{I}(x, y) - \mathbf{M}(x, y)\vec{n}(x, y)\|^2. \quad (9)$$

Many possible solutions to equation 9 could exist, however, that violate (1) the normality constraint on the recovered surface normals, $\vec{n}^T(x, y)\vec{n}(x, y) = 1$, and (2) the assumed smoothness of the recovered orientation map. To insure that each normal is unity, a new term is incorporated into the error expression yielding

$$\min_{\mathbf{M}, \vec{n}} \epsilon = \sum_{x,y} \|\vec{I}(x, y) - \mathbf{M}(x, y)\vec{n}(x, y)\|^2 + \mu(\|\vec{n}(x, y)\|^2 - 1)^2. \quad (10)$$

where μ is a scalar. It is of interest to note that Drew's method for recovering shape given unknown reflectance and lighting conditions is essentially a closed-form solution to equation 10 over both \mathbf{M} and \vec{n} .

In the presence of non-constant surface reflectance, however, an additional constraint based on surface smoothness must be added to insure a continuous solution. Because any surface from which shape can be extracted is assumed C^2 continuous, the smoothness penalty criteria,

$$\left\| \frac{\partial \vec{n}(x, y)}{\partial x} \right\|^2 + \left\| \frac{\partial \vec{n}(x, y)}{\partial y} \right\|^2 = \|\vec{n}_x(x, y)\|^2 + \|\vec{n}_y(x, y)\|^2,$$

can be used to construct the functional constraint,

$$\begin{aligned} \min_{\mathbf{M}, \vec{n}} \epsilon = & \sum_{x,y} \|\vec{I}(x, y) - \mathbf{M}(x, y)\vec{n}(x, y)\|^2 + \\ & \beta (\|\vec{n}_x(x, y)\|^2 + \|\vec{n}_y(x, y)\|^2) + \mu (\|\vec{n}(x, y)\|^2 - 1)^2 \end{aligned} \quad (11)$$

where β is a constant.

Given the shape-from-color transform, $\mathbf{M}(x, y)$, the function of equation 11 can be minimized with respect to the surface orientation map, $\vec{n}(x, y)$. Using the calculus of variations, the Euler equation corresponding to equation 11 is given by

$$\mathbf{M}^T (\vec{I} - \mathbf{M}\vec{n}) + \beta \nabla^2 \vec{n} - \mu \vec{n} = 0. \quad (12)$$



where the dependence on (x, y) has been dropped for notational simplicity. Solving equation 12 for the scaled surface orientation $\mu\vec{n}$ yields

$$\mu\vec{n} = \mathbf{M}^T (\vec{I} - \mathbf{M}\vec{n}) + \beta\nabla^2\vec{n} \quad (13)$$

Pre-multiplying equation 13 by \vec{n}^T produces the following expression for the Lagrangian multiplier, μ ,

$$\begin{aligned} \mu &= \vec{n}^T \mathbf{M}^T (\vec{I} - \mathbf{M}\vec{n}) + \beta\vec{n}^T \nabla^2\vec{n} \\ &= \vec{n}^T \mathbf{M}^T (\vec{I} - \mathbf{M}\vec{n}) + \beta(\nabla^2\vec{n} \cdot \vec{n}) \\ &= (\mathbf{M}\vec{n})^T (\vec{I} - \mathbf{M}\vec{n}) + \beta(\nabla^2\vec{n} \cdot \vec{n}) \end{aligned} \quad (14)$$

Substituting this expression for μ into equation 12 yields

$$(\mathbf{I} - \vec{n}\vec{n}^T)\mathbf{M}^T (\vec{I} - \mathbf{M}\vec{n}) + (\mathbf{I} - \vec{n}\vec{n}^T)\beta\nabla^2\vec{n} = 0$$

Letting $\mathbf{K} = (\mathbf{I} - \vec{n}\vec{n}^T) \in \mathcal{R}^{3 \times 3}$ reduces the above expression to

$$\mathbf{K} \left(\mathbf{M}^T (\vec{I} - \mathbf{M}\vec{n}) + \beta\nabla^2\vec{n} \right) = 0 \quad (15)$$

To avoid the trivial solution to the above expression where

$$\vec{n} = \mathbf{M}^T (\vec{I} - \mathbf{M}\vec{n}) + \beta\nabla^2\vec{n}$$

let the updated solution to the above expression be given by \vec{m} , such that $\vec{m} = \vec{v} + \kappa\vec{n}$ where \vec{v} is orthogonal to \vec{n} . This leads to the expression

$$\mathbf{K}\vec{m} = \mathbf{K}(\vec{v} + \kappa\vec{n}) = \mathbf{K}\vec{v} = \vec{v}$$

where κ is a scalar $= \sqrt{1 - \|\vec{v}\|^2}$. Hence, by substitution, equation(15) can be written as

$$\mathbf{K}\vec{m} = \vec{v} = \mathbf{K}\beta\nabla^2\vec{n} + \mathbf{K}\mathbf{M}^T (\vec{I} - \mathbf{M}\vec{n})$$

Using the above expressions for \mathbf{K} , \vec{m} and \vec{v} , the following sets of equations can be used to iteratively update a globally optimal surface orientation map using color image data in the presence of non-constant surface reflectance assuming the shape-from-color matrix is known:

$$\begin{aligned} \mathbf{K}^{(t)} &= \mathbf{I} - \vec{n}^{(t)}\vec{n}^{(t)T} \\ \vec{v}^{(t)} &= \mathbf{K}^{(t)} \left(\beta\nabla^2\vec{n}^{(t)} + \mathbf{M}^T (\vec{I} - \mathbf{M}\vec{n}^{(t)}) \right) \\ \vec{n}^{(t+1)} &= \vec{v}^{(t)} + \vec{n}^{(t)} \sqrt{1 - \|\vec{v}^{(t)}\|^2}. \end{aligned} \quad (16)$$

The iterative procedure is stopped when the change in the *RMS* error, $\delta\epsilon$ is less than some user specified criteria, ϵ_s .

Estimates of both the shape-from-color matrix, \mathbf{M} , and surface orientation function, $\vec{n}(x, y)$ are required to use the iterative update equations of (16). If the illumination and reflectance conditions of the scene are not known, the single color method described in section 2 can be used to recover estimates of both parameters. When the single color



estimates are employed, the variational scheme starts from a quadratically constrained solution and proceeds to a solution which is both quadratically and smoothness constrained. The residual error, ϵ , will not be zero due to the fact that the mapping, \mathbf{M} , is not actually constant for a surface with varying reflectance. However, if the multiplier, β , is chosen such that

$$\|\vec{I}(x, y) - \mathbf{M}\vec{n}^{(t)}\|^2 \ll \beta \left(\|\vec{n}_x^{(t)}\|^2 + \|\vec{n}_y^{(t)}\|^2 \right)$$

at values away from the desired solution, the process will converge to the correct estimate for the surface orientation map,

$$\vec{n}^{(t)} = \mathbf{E}^{-1}\vec{n}_e(x, y) = \vec{n}(x, y),$$

which is the closest smooth solution to the starting condition, $\vec{n}_e(x, y)$.

The complete variational approach to estimating shape from a single color image is outlined below.

Variational Shape-From-Color Algorithm

Input Parameters

- RGB Color Image - $\vec{I}(x, y)$.
- Continuity Constraint Coefficient - β .
- Minimal Error Coefficient - ϵ_s .
- (Optional) Shape-from-Color Transform - \mathbf{M} .

Process

1. If $\mathbf{M}(x, y)$ is not specified:
 - Construct the linear system 6 from the elliptical constraint equation 5,

$$\mathbf{F}\vec{z} = 1,$$

and solve for \vec{z} .

- Construct the matrix, $\mathbf{\Gamma} = \mathbf{M}^{-T}\mathbf{M}^{-1}$, from the entries of \vec{z} .
 - Compute an estimate of the inverse shape-from-color transform, \mathbf{M}^{-1} from $\mathbf{\Gamma}$ using Cholesky decomposition.
 - Invert \mathbf{M}^{-1} to determine the shape-from-color matrix estimate, \mathbf{M} .
2. Set the iteration counter, $t = 0$.
 3. Compute the initial orientation estimate for each surface point,

$$\vec{n}^{(t)} = \mathbf{M}^{-1}\vec{I}(x, y).$$

4. Compute the initial residual error,

$$\epsilon^{(t)} = \sum_{x,y} \|\vec{I}(x, y) - \mathbf{M}\vec{n}^{(t)}\|^2 + \beta \left(\|\vec{n}_x^{(t)}\|^2 + \|\vec{n}_y^{(t)}\|^2 \right) + \mu \left(\|\vec{n}^{(t)}\|^2 - 1 \right)^2.$$

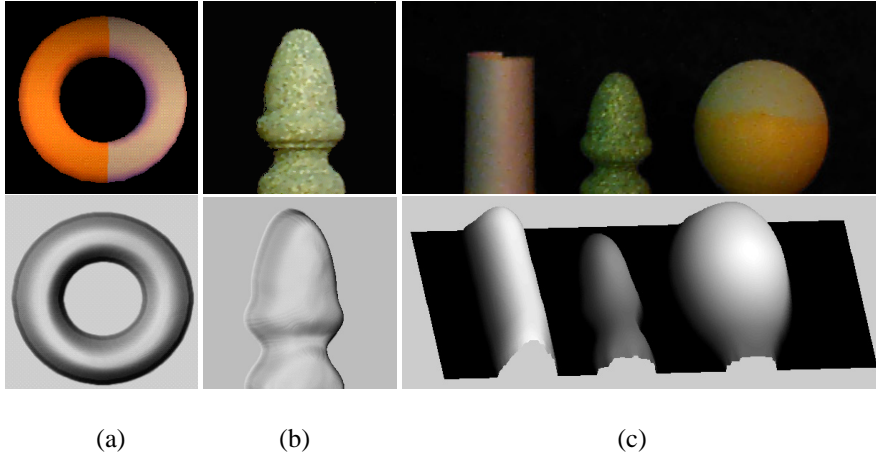


Figure 2: Examples of simple shapes recovered using the variational shape-from-color approach described in section 3.

5. Update the orientation estimate for each surface point using the expressions

$$\begin{aligned}\mu^{(t)} &= (\mathbf{M}\vec{n}^{(t)})^T(\vec{I} - \mathbf{M}\vec{n}^{(t)}) + \beta(\nabla^2\vec{n}^{(t)} \cdot \vec{n}^{(t)}) \\ \mathbf{K}^{(t)} &= \mathbf{I} - \vec{n}^{(t)}\vec{n}^{(t)T} \\ \vec{v}^{(t)} &= \mathbf{K}^{(t)} \left(\beta\nabla^2\vec{n}^{(t)} + \mathbf{M}^T(\vec{I} - \mathbf{M}\vec{n}^{(t)}) \right) \\ \vec{n}^{(t+1)} &= \vec{v}^{(t)} + \vec{n}^{(t)}\sqrt{1 - \|\vec{v}^{(t)}\|^2}.\end{aligned}$$

6. Compute the residual error

$$\epsilon^{(t+1)} = \sum_{x,y} \|\vec{I}(x,y) - \mathbf{M}\vec{n}^{(t+1)}\|^2 + \beta \left(\|\vec{n}_x^{(t+1)}\|^2 + \|\vec{n}_y^{(t+1)}\|^2 \right) + \mu \left(\|\vec{n}^{(t+1)}\|^2 - 1 \right)^2.$$

7. If $\delta\epsilon = \epsilon^{(t+1)} - \epsilon^{(t)} < \epsilon_s$ then exit, else increment the iteration counter, $t = t + 1$, and return to step 5.

Output Data

- Surface Orientation Map - $\vec{n}(x,y) = \vec{n}^{(t)}$.

4 Experimental Results

The ability of the variational algorithm to recover both qualitatively and quantitatively accurate shape reconstructions under unknown illuminants and in the presence of non-constant surface reflectance has been studied using both synthetic and real images. Figure 2 shows the surface shapes constructed by applying the variational shape-from-color

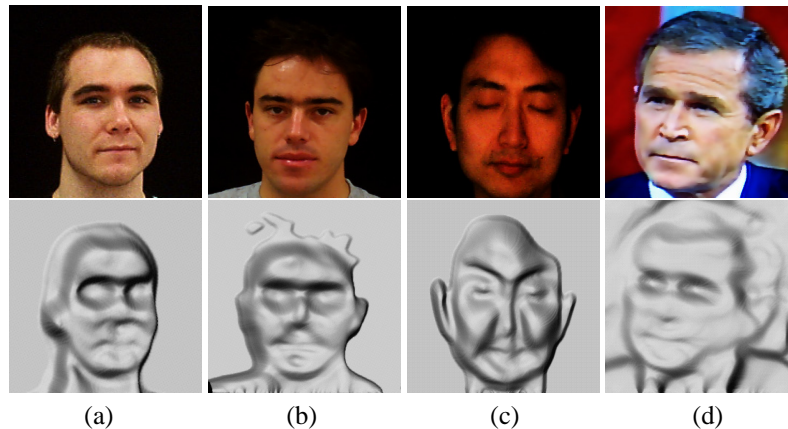


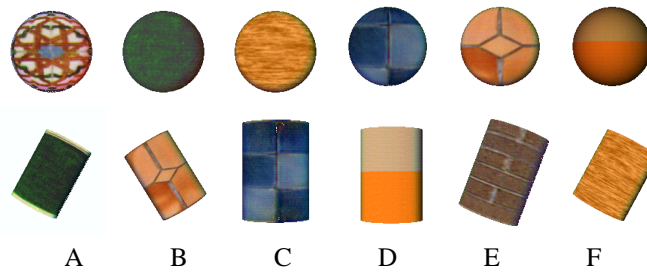
Figure 3: Examples of facial shapes recovered using the variational shape-from-color algorithm described in section 3.

algorithm to both synthetic and real images of simple objects. Note the qualitatively high quality of the recovered surfaces across the range from simple to complex color textures. Examples of surfaces recovered by applying the iterative algorithm to images of human faces are provided in figure 3. While the visual quality of the surfaces vary due to the presence of unmodeled illumination artifacts, such as self-cast shadows and specular regions, they are still easily linked to their generating image, even for the case shown in figure 3(d) where the image was photographed from a television monitor.

Figure 4 presents results from two sets of experiments designed to quantify the ability of the variational surface estimation technique. First, shapes were constructed using the iterative method from images of spheres and cylinders, and quadratic functions were then fit to these recovered shapes. The LMS error in pixel units between the best functional representation and the actual surface points was then computed as a quantitative measurement of the recovered surface accuracy. These tests were performed using both the original images and those corrupted by salt and pepper noise with density $\sigma = 0.3$. Note the low average LMS error rates of 0.066 and 0.135 normalized units in the uncorrupted sphere and cylinder tests, respectively; and the robustness of the variational approach to the presence of random image noise.

5 Concluding Remarks

Previously reported shape estimation algorithms that employ color images assume smooth surfaces with constant surface reflectance, limiting their applicability to images of most real-world objects. Through explicit use of a continuity enforcement constraint, the variational approach to color-image-based shape estimation presented in this paper places no restrictions on the visible surface texture. Thus, the new technique is able to recover the shape of any Lambertian reflector given only a single color image.



LMS Error - Spheres							
Noise	Image A	Image B	Image C	Image D	Image E	Image F	Average
$\sigma = 0.0$	0.055	0.086	0.079	0.054	0.067	0.058	0.066
$\sigma = 0.3$	0.075	0.071	0.082	0.064	0.048	0.062	0.067
LMS Error - Cylinders							
$\sigma = 0.0$	0.133	0.170	0.053	0.073	0.201	0.185	0.135
$\sigma = 0.3$	0.166	0.179	0.065	0.072	0.219	0.190	0.148

Figure 4: Quantitative evaluation of the variational shape-from-color algorithm described in section 3. The table shows the per-test and average LMS difference between the best-fit conic and the computed shape for the images shown.

References

- [1] K.E Torrance and E.M. Sparrow, “Theory for offspecular reflection from roughened surface for ray reflection,” *Journal of Optical Society of America A*, vol. 65, no. 1, pp. 531–536, 1967.
- [2] K. Ikeuchi and B.K. Horn, “Numerical shape from shading and occluding boundaries,” *Artificial Intelligence*, vol. 17, pp. 141–184, 1981.
- [3] A. Pentland, “Linear shape from shading,” *International Journal of Computer Vision*, vol. 4, pp. 153–162, 1990.
- [4] T. Kanade, “Recovery of the three-dimensional shape of an object from a single view,” *Artificial Intelligence*, vol. 17, pp. 409–460, 1981.
- [5] W. Zhao and R. Chellappa, “Symmetric shape-from-shading using self-ratio image,” *International Journal of Computer Vision*, vol. 45, no. 1, pp. 55–75, October 2001.
- [6] M. Drew and L. Kontsevich, “Closed-form attitude determination under spectrally varying illumination,” *IEEE International Conference on Computer Vision and Pattern Recognition*, pp. 985–990, 1994.
- [7] L. Kontsevich, A. Petrov, and I. Vergelskaya, “Reconstruction of shape from shading in color images,” *Journal of Optical Society of America A*, 1994.
- [8] G. Golub and F Van Loan, *Matrix Computations*, The John Hopkins University Press, 2nd edition, 1989.