

The Cross ratio: A Revisit to its Probability Density Function

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Abstract

The cross ratio has wide applications in computer vision because of its invariance under projective transformation. In active vision where the projections of quadruples of collinear landmark points in the scene are tracked in the image sequence for robot localisation or online camera calibration, one often needs to compute cross ratios from noisy image data for some subsequent operations. Being able to assess the reliability of each computed cross ratio value against a known level of image noise is therefore of importance. This aim motivates our research to derive the probability density function (p.d.f.) of the cross ratio based on the normality assumption of the associated random variables and to investigate into empirical cases where this assumption fails to hold. Although an analytical formula for the general p.d.f. of the cross ratio has not been achieved, our research results show that (i) the distance between the closest pair of collinear points is a significant factor that determines the shape of the p.d.f. of the cross ratio and (ii) a good estimate of the cross ratio can be obtained if the points of the quadruple are sufficiently far apart.

Keywords: cross ratio, projective invariant, density function, simulation, frequency histogram.

1 Introduction

The study of projective invariants has drawn much attention of computer vision researchers in recent years. Applications of projective invariants include the use of cross ratios [11] in calibrating structured light stripe systems [3, 5] and model-based recognition [7] and the use of coplanar point configuration [10, 2] for recognition tasks. Theoretical work on the cross ratio includes the reports on the probability density function (p.d.f.) of the cross ratio initiated by Åström and Morin [1] and Maybank [8]. The derivations reported in both of these research works are based on the assumption that the four collinear points for the cross ratio are independent, identically distributed random variables. Åström and Morin use the uniform distribution for the four collinear points and compute the p.d.f. via Maple; Maybank uses the normal distribution for the four collinear points and derives the p.d.f. analytically in terms of elliptic integral. Both density functions have the same shape, with symmetry about the cross ratio value 0.5 and singularities at the cross ratio values

0, 1, and $\pm\infty$. Thus, if the notation $\mathcal{N}(\mu, \sigma^2)$ denotes the Gaussian random variable of mean μ and variance σ^2 then, in Maybank's derivation, all the points of the quadruple are $\mathcal{N}(0, 1)$. If the notation $\mathcal{U}(a, b)$ denotes the uniform random variable over the interval $[a, b]$ then, in Åström and Morin's derivation, all the points of the quadruple are $\mathcal{U}(0, 1)$.

In this paper, we revisit the probability density function of the cross ratio with a different assumption on the collinear points of the quadruple. We assume that the 4 collinear points are independent Gaussian random variables of the same variance but different means. That is, each point of the quadruple is a $\mathcal{N}(\mu_i, \sigma^2)$ random variable such that $\mu_i \neq \mu_j$ if $i \neq j$. We will show that, when the means of the 4 Gaussian random variables of the quadruple are not identical, the probability density function of the cross ratio is very different from that derived in [1, 8]: in fact, the shape of the p.d.f. of the cross ratio is governed by the distance between the closest pair of image points that constitute the quadruple.

We organise the paper as follows. Section 2 describes the derivation of the probability density function of the cross ratio. Section 3 compares the derived p.d.f. with the frequency histogram obtained from simulation. Section 4 discusses a few related issues and concludes the paper.

2 Probability density function of the cross ratio

2.1 Intermediate random variables and assumptions

Let the four collinear points be independent Gaussian random variables Θ_i , $1 \leq i \leq 4$, of mean μ_i and variance σ^2 respectively, denoted by the notation $\Theta_i \sim \mathcal{N}(\mu_i, \sigma^2)$. Before we proceed to define the cross ratio T , let us define two intermediate random variables Z and V as follows:

$$Z = (\Theta_1 - \Theta_3)(\Theta_2 - \Theta_4) \quad (1)$$

$$V = (\Theta_1 - \Theta_4)(\Theta_2 - \Theta_3). \quad (2)$$

Using the independence property of the Θ_i 's, it is straightforward to verify that $\Theta_1 - \Theta_3$ and $\Theta_2 - \Theta_4$ are independent Gaussian random variables with means $\mu_1 - \mu_3$ and $\mu_2 - \mu_4$ and variance $2\sigma^2$. However, the distributions of Z (and likewise for V) are not well-defined analytically.

The expected values, μ_Z and μ_V , the variances, σ_Z^2 and σ_V^2 , and the covariance, σ_{ZV} , of Z and V are computed as

$$\begin{aligned} \mu_Z &= (\mu_1 - \mu_3)(\mu_2 - \mu_4) \\ \sigma_Z^2 &= 2\sigma^2 \left((\mu_2 - \mu_4)^2 + (\mu_1 - \mu_3)^2 + 2\sigma^2 \right) \\ \mu_V &= (\mu_1 - \mu_4)(\mu_2 - \mu_3) \\ \sigma_V^2 &= 2\sigma^2 \left((\mu_1 - \mu_4)^2 + (\mu_2 - \mu_3)^2 + 2\sigma^2 \right) \\ \sigma_{ZV} &= \mathbb{E} \left((Z - \mu_Z)(V - \mu_V) \right) \\ &= \mathbb{E} \left((\Theta_1 - \Theta_3)(\Theta_2 - \Theta_4)(\Theta_1 - \Theta_4)(\Theta_2 - \Theta_3) \right) - \mu_Z \mu_V \\ &= (\sigma^2 + \mu_1^2)((\sigma^2 + \mu_2^2) - \mu_2 \mu_3 - \mu_2 \mu_4 + \mu_3 \mu_4) + (\sigma^2 + \mu_2^2)(-\mu_1 \mu_3 - \mu_1 \mu_4 + \mu_3 \mu_4) + \\ &\quad (\sigma^2 + \mu_3^2)(\mu_1 \mu_2 - \mu_1 \mu_4 - \mu_2 \mu_4 + (\sigma^2 + \mu_4^2)) + (\sigma^2 + \mu_4^2)(\mu_1 \mu_2 - \mu_1 \mu_3 - \mu_2 \mu_3) + \\ &\quad 2\mu_1 \mu_2 \mu_3 \mu_4 - \mu_Z \mu_V, \end{aligned}$$

where $E(\cdot)$ denotes the expectation of the entity concerned. In the estimations above, the independence property of Θ_i 's and the expression $E(\Theta_i^2) = \sigma^2 + \mu_i^2$ have been employed.

2.2 A derivation

The cross ratio T of the Θ_i 's is then defined by

$$T = Z/V, \quad (3)$$

the quotient of two random variables. If the joint p.d.f., $f_{Z,V}$, of Z and V is known, then the p.d.f. of T is simply

$$q_T(\tau) = \int_{-\infty}^{\infty} |v| f_{Z,V}(\tau v, v) dv, \quad (4)$$

where we have used the definition $\tau = z/v$.

However, the joint distribution $f_{Z,V}$ is not known because f_Z and f_V cannot be derived analytically except for the trivial case where $\mu_i = 0$, for all i . In this paper, we will not study the p.d.f. of the cross ratio when all the μ_i 's are zero. Interested reader is referred to [1] when $\Theta_i \sim \mathcal{U}(0, 1)$ and [8] when $\Theta_i \sim \mathcal{N}(0, 1)$, for all i .

In the remaining of the paper, we will assume $f_{Z,V}$ to be a Gaussian distribution. The subsequent derivation of the p.d.f. of T is therefore valid only if $f_{Z,V}$ can be approximated by a Gaussian distribution. Cases where this assumption does not hold will be discussed later in the paper.

Let Λ be the covariance matrix of Z and V . The joint probability density function for Z and V is then given by

$$f_{Z,V}(z, v) = \frac{1}{2\pi|\Lambda|^{1/2}} \exp \left\{ -\frac{1}{2} \begin{pmatrix} z - \mu_Z & v - \mu_V \end{pmatrix} \Lambda^{-1} \begin{pmatrix} z - \mu_Z \\ v - \mu_V \end{pmatrix} \right\}. \quad (5)$$

Further simplification to (4) is necessary. Substituting (5) into (4) and collecting terms that contain v gives

$$\begin{aligned} q_T(\tau) &= \frac{\exp(-k_0)}{2\pi|\Lambda|^{1/2}} \int_{-\infty}^{\infty} |v| \exp \{ -k_2 v^2 - k_1 v \} dv \\ &= \frac{\exp(-k_0)}{2\pi|\Lambda|^{1/2}} \left(\int_0^{\infty} v \exp \{ -k_2 v^2 - k_1 v \} dv + \int_0^{\infty} v \exp \{ -k_2 v^2 + k_1 v \} dv \right) \end{aligned} \quad (6)$$

where

$$\begin{aligned} k_2 &= \frac{1}{2} (S_{11}\tau^2 + 2S_{12}\tau + S_{22}) \\ k_1 &= -S_{11}\mu_Z\tau - S_{12}\mu_V\tau - S_{12}\mu_Z - S_{22}\mu_V \\ k_0 &= \frac{1}{2} (S_{11}\mu_Z^2 + 2S_{12}\mu_Z\mu_V + S_{22}\mu_V^2) \\ S &\equiv [S_{ij}] \equiv \Lambda^{-1}. \end{aligned}$$

Matrix S given in the expressions above is the information matrix, or the inverse covariance matrix, of Z and V .

The above integral exists only if $k_2 > 0$. From the definition of k_2 , it reveals that $k_2 = \frac{1}{2}(\tau - 1)S(\tau - 1)^\top$. Since S is positive definite, k_2 is positive for all values of τ . The existence of the integral in (6) is thus guaranteed.

One may proceed with the integration by part technique to reduce (6) further. Alternatively, a useful formula can be culled from [4] (page 382, 3.462 (1)) and the integral in (6) is reduced to

$$q_T(\tau) = \frac{\exp(-k_0)}{4\pi k_2 |\Lambda|^{1/2}} \exp\left(\frac{k_1^2}{8k_2}\right) \left(D_{-2}\left(k_1/\sqrt{2k_2}\right) + D_{-2}\left(-k_1/\sqrt{2k_2}\right) \right) \quad (7)$$

where D_{-2} is the parabolic cylinder function given by

$$D_{-2}(x) = \exp(x^2/4) \sqrt{\pi/2} \left\{ \sqrt{\pi/2} \exp(-x^2/2) - x \left(1 - \operatorname{erf}\left(x/\sqrt{2}\right) \right) \right\}$$

and erf is the error function defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.$$

2.3 Symmetries of the distribution

If one takes into consideration all the 24 permutations of the parameters (i.e. the Θ_i 's) of the four collinear points when computing the cross ratio, then 6 different values of the cross ratio arise, each of which has equal probability of occurrences and falls respectively onto one of the following disjoint regions: $\tau \leq -1$, $-1 \leq \tau \leq 0$, $0 \leq \tau \leq \frac{1}{2}$, $\frac{1}{2} \leq \tau \leq 1$, $1 \leq \tau \leq 2$, $\tau \geq 2$ (see [8] for a proof).

The p.d.f. of the cross ratio T given in (7) is for one of these permutations only. Let $q_T^{(i)}$, for $1 \leq i \leq 6$, be the six computed p.d.f.'s using the formula given in (7) for the six different cross ratios from permutations. The p.d.f. of the cross ratio T is then

$$f_T(\tau) \equiv \frac{1}{6} \sum_{i=1}^6 q_T^{(i)}(\tau). \quad (8)$$

3 Verification of the p.d.f. of T using simulated data

To verify the validity of the derived p.d.f. given in (8), experiments have been conducted using simulation data. Arbitrary mean values μ_i , for $1 \leq i \leq 4$, of the random variables Θ_i 's were input by the user. The variance σ^2 was set to 1 for all the Θ_i 's. Other values of σ^2 can also be used; however, in frequency histogram generation for the cross ratio, increasing/decreasing σ^2 is equivalent to decreasing/increasing the relative distance between the mean values μ_i 's. Hence, we varied the means μ_i 's while keeping σ^2 fixed in all our simulations.

10^6 random values for each random variable Θ_i were synthesized. These values were then used to compute the values for the Z and V random variables using formulae (1)–(2) and finally we obtained 10^6 different cross ratio values for T using (3). Incorporating the permutation of points of the quadruple, we obtained 6×10^6 cross ratio values. The formula given in (7) for each of the six q_T 's and the final p.d.f. of the cross ratio given in (8) were coded also in Matlab. To compare the computed f_T with the frequency histogram

of T from simulation, the area under the histogram of T was estimated using Simpson's Rule. Since the f_T obtained from (8) has its area under the curve being 1, a simple scaling of f_T will show whether the histogram and f_T coincide.

3.1 Experiments

Fig. 1(a) shows the 6 frequency histograms of the cross ratios arisen from the permutations of the four Θ_i 's whose means are $-10, -5, 3, 20$ respectively. The computed

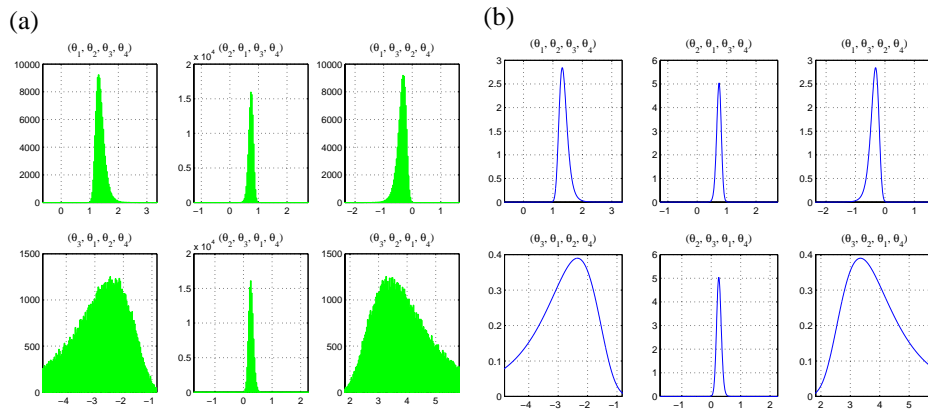


Figure 1: (a) Histograms of the six cross ratios of $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ for $(\mu_1, \mu_2, \mu_3, \mu_4) = (-10, -5, 3, 20)$. (b) The computed p.d.f. of each of the corresponding cross ratios.

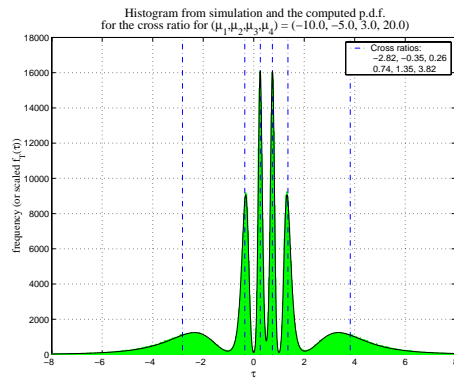


Figure 2: Simulation (green shaded region) and the computed p.d.f. (solid black line) of the cross ratio for the means $(\mu_1, \mu_2, \mu_3, \mu_4) = (-10, -5, 3, 20)$.

p.d.f. corresponding to each of these cross ratios is given in Fig. 1(b). The six histograms and the six computed p.d.f. have very similar shape except that their vertical axes have different scales. The combined p.d.f. and the simulation are shown in Fig. 2. The computed p.d.f. corresponds very well with the frequency histogram of T . Note that the shape

of the p.d.f. is very different from those obtained by [1, 8]. The p.d.f. of T in this example has six peaks, four of the central ones correspond well to the four cross ratios (marked as vertical dash-dotted blue lines) about $\tau = \frac{1}{2}$. However, the p.d.f. of T of a different set of μ_i values has a very different shape. For instance, Fig. 3 shows that as distances among the collinear points decrease (i.e. the distances among the μ_i 's decrease), the central peaks merge. Note that in Fig. 3, the distance between the closest pair of μ_i 's is $2 - (-1) = 3$, while in Fig. 2, this distance is $5 - (-10) = 5$.

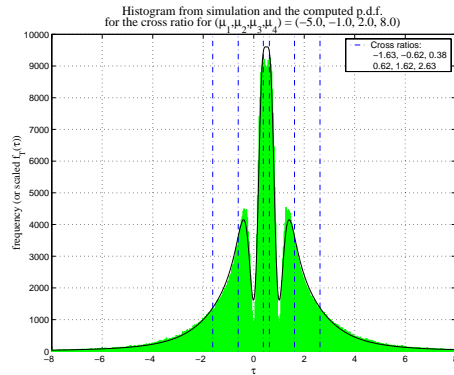


Figure 3: Simulation (green shaded region) and the computed p.d.f. (solid black line) for the cross ratio for the means $(\mu_1, \mu_2, \mu_3, \mu_4) = (-5, -1, 2, 8)$.

Decreasing further the difference among the mean values, μ_i 's, sees the merging of the peaks into two: one at $\tau = 0$ and another at $\tau = 1$, even though all the μ_i 's are distinct (Fig. 4(a)). In this example, the quadruple has $(\mu_1, \mu_2, \mu_3, \mu_4) = (1, 2, 5, 8)$ and the distance between the closest pair of image points is $2 - 1 = 1$.

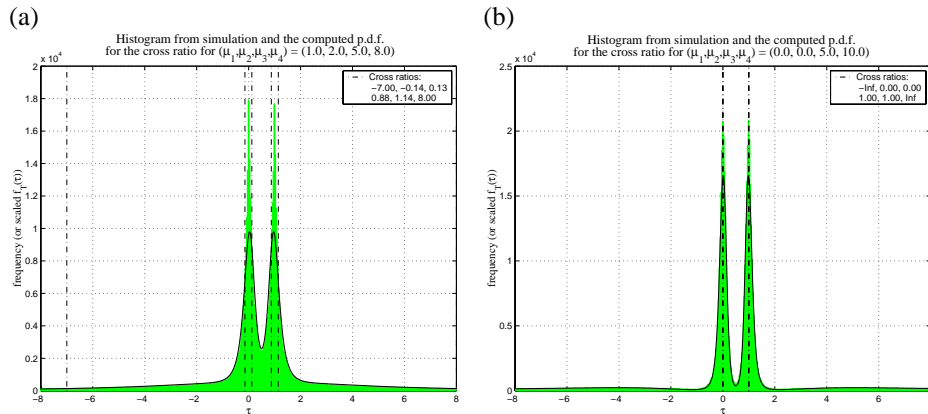


Figure 4: (a) Simulation (green shaded region) and the computed p.d.f. (solid black line) for the cross ratio for the means $(\mu_1, \mu_2, \mu_3, \mu_4) = (1, 2, 5, 8)$. (b) Simulation (green shaded region) and the computed p.d.f. (solid black line) of the cross ratio for the means $(\mu_1, \mu_2, \mu_3, \mu_4) = (0, 0, 5, 10)$.

When the quadruple has two points coincident: $\mu_1 = \mu_2 = 0$, the frequency histogram has the shape as that derived by [1, 8] (Fig. 4(b)). In this figure, the fitting of f_T to the histogram is not perfect. In fact, due to the assumption on the normality of f_Z , f_V , and $f_{Z,V}$, the computed p.d.f. always gives more gentle and shorter peaks, rather than singularities.

3.2 Non-normality of $f_{Z,V}$

We mentioned in Section 2.2 that the derived f_T given in (8) applies only if the assumption on the normality of $f_{Z,V}$ holds. While it is not trivial to assess the distribution $f_{Z,V}$ for any arbitrary μ_i values, simulations help empirically identify cases where $f_{Z,V}$ is non-normal.

Consider the following sets of mean values:

$$\begin{aligned} \text{Set 1: } & \mu_1 = -1, \quad \mu_2 = -0.5, \quad \mu_3 = 0.3, \quad \mu_4 = 2 \\ \text{Set 2: } & \mu_1 = -10, \quad \mu_2 = -5, \quad \mu_3 = 3, \quad \mu_4 = 20 \end{aligned}$$

If we consider the above as the true mean values then, under the noise-free condition, both sets will yield the same set of 6 cross ratio values. In the presence of noise (i.e. the variance $\sigma^2 \neq 0$), however, the p.d.f.'s of T for both sets are very different. Fig. 5 shows the scatter plot of V versus Z for one of the 6 cross ratios in sets 1 and 2. The shape

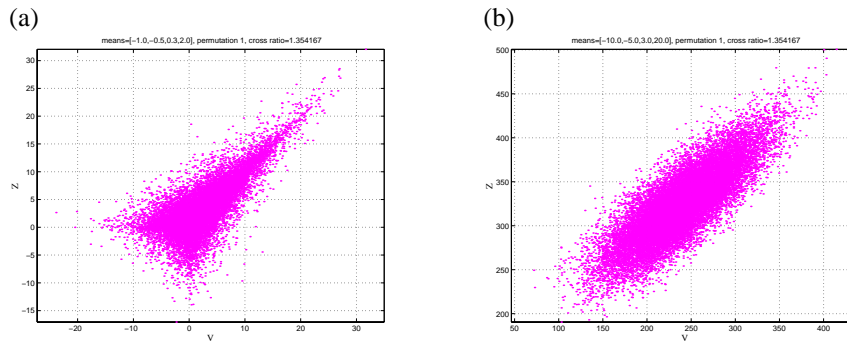


Figure 5: The scatter plot of V versus Z for (a) $(\mu_1, \mu_2, \mu_3, \mu_4) = (-1, -0.5, 0.3, 2)$, and (b) $(\mu_1, \mu_2, \mu_3, \mu_4) = (-10, -5, 3, 20)$.

formed by the points (V, Z) in set 1 is almost triangular, while the shape in set 2 can be approximated by an ellipse. The vertices of the triangle are approximately at the coordinates $(20, 20)$, $(-10, 0)$, and $(0, -10)$, corresponding to the cross ratios 1, 0, and $-\infty$ respectively. It is clear that the p.d.f. of $f_{Z,V}$ in set 1 is not normal. The non-normality of $f_{Z,V}$ is, of course, due to the non-normality of f_Z and f_V , so studying the non-normality of $f_{Z,V}$ is sufficient for the evaluation of f_T . For set 1, the f_T function, as expected, does not fit at all to the frequency histogram generated from simulation data (Fig. 6). Note also that frequency histograms shown in Figs. 6 and 2 are very different: The high level of noise ($\sigma^2 = 1$) for relatively small differences among the mean values, μ_i 's, in set 1 has drastically changed the shape of the frequency histogram. For the other permutations of the quadruple, the scatter plots for the two sets have the shapes as shown in Fig. 5 except

that the orientation of the point cluster vary according to the cross ratio value: the slope of the major axis of the point cluster is approximately equal to the absolute value of the cross ratio corresponding to that permutation.

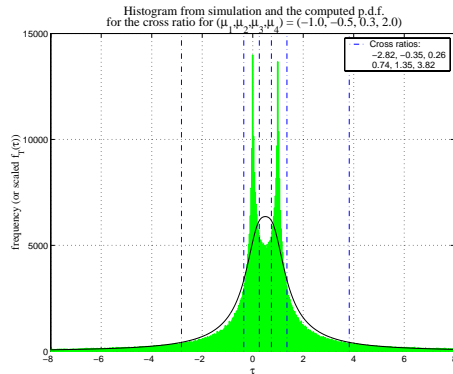


Figure 6: Simulation (green shaded region) and the computed p.d.f. (solid black line) of the cross ratio for the means $(\mu_1, \mu_2, \mu_3, \mu_4) = (-1, -0.5, 0.3, 2)$.

When the mean values μ_i 's of the four collinear points are close to each other relative to the standard deviation σ , the frequency histogram of T converges to the shape derived in [1, 8] even though the 4 image points are distinct. This shows that, when the image points are close together relative to the measure of spread σ , then statistically it is difficult to distinguish whether the four points are distinct or whether two (or more¹) of them are coincident.

3.3 Estimating the expected value and variance of T

Given that we have $\{\Theta_i \mid 0 \leq i \leq 4\}$ as the quadruple of collinear image point measurements detected from a noisy image. Let μ_i be the true image measurement of Θ_i . Assuming that the perturbation to each Θ_i follows a Gaussian distribution of zero mean and variance σ^2 , then $\Theta_i \sim \mathcal{N}(\mu_i, \sigma^2)$. We are interested in assessing the reliability of the cross ratio T computed from the quadruple $\{\Theta_i \mid 0 \leq i \leq 4\}$. Using the Taylor expansion one can obtain the following approximate formulae for the expectation and variance of T :

$$\begin{aligned}
 E(T) &\approx \frac{\mu_Z}{\mu_V} - \frac{1}{\mu_V^2} \sigma_{ZV} + \frac{\mu_Z}{\mu_V^3} \sigma_V^2 \\
 \text{var}(T) &\approx \left(\frac{\mu_Z}{\mu_V}\right)^2 \left(\frac{\sigma_Z^2}{\mu_Z^2} + \frac{\sigma_V^2}{\mu_V^2} - \frac{2\sigma_{ZV}}{\mu_Z \mu_V}\right). \tag{9}
 \end{aligned}$$

Analysis in the previous subsections shows that the formulae (9) above are biased when the skewness (the 3rd order moment) and kurtosis (the 4th order moment) [9, 6] of T are large. Indeed, Table 1 shows that, when these two terms are large, the corresponding \bar{T} and $E(T)$ are very different from the value of $\bar{\tau}$ given in the second column and that the variance of T is large accordingly. In active vision applications that involve the use of

¹The cross ratio does not exist when more than two points of the quadruple coincide.

the cross ratio for online camera calibration (e.g. using multiple images of coplanar and collinear points [12]) and model-based recognition [7], if a cross ratio cannot be confidently computed for a given level of image noise then the best solution is to zoom into the scene for a better view of the pattern concerned. This is equivalent to the analysis in this paper where $E(T)$ is closer to $\bar{\tau}$ with small $\text{var}(T)$ if the distance between the closest pair of points of the quadruple increases (and distances between other pairs of points increase proportionally). Even though the approximation stops at the second order term, the variance of T given in (9) appears to be sufficiently good for assessing the accuracy of computed cross ratio. The challenge here is to reverse the operation above — given the noise level σ^2 and the required value for $\text{var}(T)$, derive the minimum distance required between the closest pair of image points of the quadruple. This piece of information can then be fed back to the active vision system to adjust its zoom or viewing distance to the collinear points in the scene.

$(\mu_1, \mu_2, \mu_3, \mu_4)$	$\bar{\tau}$	$E(T)$	\bar{T}	$\text{var}(T)$	$\text{var}_s(T)$
(1, 2, 3, 4)	1.3333	2.8889	1.0225	1.6296	$> 10^4$
(10, 20, 30, 40)	1.3333	1.3452	1.3465	0.0099	0.0119
(-1, -0.5, 0.3, 2)	1.3542	3.9731	0.8502	3.1328	$> 10^4$
(-10, -5, 3, 20)	1.3542	1.3745	1.3771	0.0212	0.0279
(-5, -1, 2, 8)	1.6154	1.8281	1.6192	0.2432	2719.3203
(0, 0, 1, 2)	1.0000	1.7500	18.4668	1.5000	$> 10^4$
(0, 0, 5, 10)	1.0000	1.0108	1.0198	0.0216	0.2461
(0, 0, 10, 50)	1.0000	1.0064	1.0070	0.0128	0.0142

Table 1: Column 1 shows the mean values of image points that constitute the quadruple; column 2 shows the true cross ratio $\bar{\tau}$ in the absence of image noise where $\bar{\tau} = ((\mu_1 - \mu_3)(\mu_2 - \mu_4))/((\mu_1 - \mu_4)(\mu_2 - \mu_3))$; column 3 shows the expected value of T as computed using formula (9); column 4 shows the mean, \bar{T} , of 10^6 simulated cross ratio values; column 5, $\text{var}(T)$, shows the estimated variance using formula (9); the last column $\text{var}_s(T)$ shows the variance of T from the simulated cross ratio values.

4 Discussions and conclusions

An analytical form of the p.d.f. of the cross ratio has not been achieved because of the complexity involved in the non-normality of the associated random variables Z and V . Only when the distances between the adjacent pairs of image points of the quadruple are sufficiently large for the noise level σ^2 is the derived p.d.f. of T a good approximation of the frequency histogram from simulation. We note the following: when any two of the mean values μ_i 's coincide, the frequency histogram of T has singularities at 0, 1, and $\pm\infty$; the derived f_T , which may give a reasonable fit to the frequency histogram (see Fig. 4(b)), has no singularities but only sharp peaks. A study on formula (7) quickly reveals that the derived q_T gives no singularities.

Instead of assuming that the Θ_i 's are independent Gaussian random variables $\mathcal{N}(\mu_i, \sigma^2)$,

an alternative is to assume that they are independent uniform random variables over different intervals which are disjoint, partially or totally overlapped. It is also of interest to estimate the probability of rejection and false alarms for model-based recognition [7, 8]. However, since the p.d.f. derived in this paper only applies to cases where the distribution of Z and V can be approximated by a Gaussian, a more general p.d.f. of the cross ratio T must be first estimated.

We have shown in this paper a derivation of the p.d.f. of the cross ratio based on the assumption that the points of the quadruple are independent Gaussian random variables of different means. This assumption is different from that used in [8] where the mean values of all the points of the quadruple are identical. By letting the points of the quadruple be Gaussian random variables of different means, our derivation and simulations have shown that the shape of the p.d.f. depends on the minimum distance of the adjacent pairs of the 4 collinear points and that a good estimate of the cross ratio can only be obtained when this distance is sufficiently large with respect to the known level of image noise.

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