

Projective Geometry Based Image Reconstruction: Limitations and Applicability Constraints

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Abstract

The 3D image reconstruction method based on Projective Geometry is very attractive for practical applications because it does not require any camera calibration. It requires only the knowledge of at least 8 reference points defining two planes. We investigate here the sensitivity of the results of this method to the measurement errors in the positions of the reference points both on the image plane and in the 3D world, and suggest some procedures which should be used in the practical applications of the method to avoid excessive error amplification.

1 Introduction

One of the objectives of computer vision is the recovery of the 3-D information lost by the process of recording a scene on the 2-D image plane. This information cannot be recovered in general from one conventional intensity image only. At least two images are needed obtained either by the same camera and exploiting the principles of motion parallax, or obtained by two stationary cameras and exploiting the principles of stereo vision.

We are interested here in the latter approach and in particular in the problem of recovering the 3-D shape of a block stone. This is part of a major project concerned with the optimal cutting of a block of granite into slabs to achieve minimum waste. Each block is placed on a specially constructed platform and viewed from four cameras around it, placed at approximately 90° angular distance from each other (figure 1). Each camera "sees" only three planes of the stone which has approximately the shape of a parallelepiped. The top plane is viewed by all four cameras but any side plane is viewed by only two adjacent cameras. So, the problem we are interested in, is effectively trying to reconstruct the equation of a plane which is viewed by two cameras, call them left and right cameras, assuming that the correspondence problem has somehow been solved.

There are basically two methods proposed in the literature: one is based on Projective Geometry [5], and the other on Camera Calibration [6]. The process of camera calibration, however, is not very practical in the factory floor where blocks of granite are moved about by cranes. An efficient and robust method is needed which will not require the use of any calibration devices. Projective geometry approaches to 3-D vision have been used before

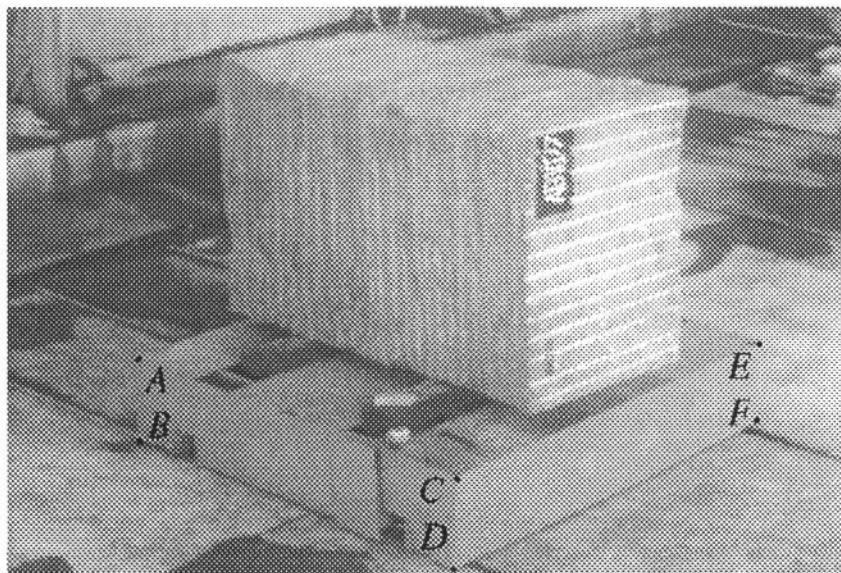


Figure 1: A granite block to be reconstructed in 3D is placed on a platform.

for the recovery of planar surfaces [1] and for reconstruction without using any calibration parameters [3]. However, none of these papers includes a detailed sensitivity analysis.

The Projective Geometry approaches rely on the knowledge of 8 reference points which define two planes in 3D. A good option offered to us is that of using the platform on which the stone stands. The platform is a robust structure (designed to support tones of granite) and can be made to have some desired characteristics like to be a right angled parallelepiped with well defined vertices, and faces painted with distinct colours for easy identification. Each camera can be made to see two faces of the platform for which the vertices can be easily identified. The world coordinate position of each vertex is assumed to be known. Thus in each image, we shall have a set of 6 reference points (points A, B, C, D, E and F in figure 1) forming two planes intersecting along a line in the image. The method of Projective Geometry can therefore be employed for the 3D reconstruction of the granite block. The user requirement, however, is that the reconstruction error should not exceed 1% in terms of the linear dimensions of the block. Given that there is always some inaccuracy in the estimation of the location of vertices in an image and certainly inaccuracy in the construction of the platform (in terms of which the world coordinate system is defined) and the measurement of its vertices (the reference points) it is very important to check whether the adopted method can be relied upon to cope with uncertainties.

Detailed experimentation we performed with settings where ground truth was known, showed that there were cases where the error exceeded 2000%! This motivated us to look more carefully at the proposed method. In this paper we show the error analysis performed for the various stages of the Projective Geometry method and the limitations to its applicability dictated by this error analysis. In section 2 we shall describe briefly the method as proposed by Mohr [5] and in section 3 we shall present our error analysis. We shall conclude in section 4.

2 Projective Geometry Based 3D Reconstruction

2.1 Introduction to Projective Geometry

If A, B, C and D are four co-linear points then their *cross-ratio* is defined as:

$$[A, B, C, D] = \frac{\overline{CA}}{\overline{CB}} \cdot \frac{\overline{DB}}{\overline{DA}} \quad (1)$$

where $\overline{AB}$ is the *directed Euclidean distance* of the two points A and B . This means that $\overline{AB} = -\overline{BA}$. If the barycentric representation of the line is used, i.e.: $\vec{b} + \mu\vec{d} = \vec{r}$, where $\vec{r}$ is the position vector of any point along the line, μ is a parameter taking real values and $\vec{b}$ and $\vec{d}$ are the base and directional vectors of the line respectively, then the cross-ratio can be expressed in terms of the μ value of every point in the line l , that is:

$$[A, B, C, D] = \frac{\mu_C - \mu_A}{\mu_C - \mu_B} \cdot \frac{\mu_D - \mu_B}{\mu_D - \mu_A} \quad (2)$$

where μ_A is the value of the parameter μ in the equation of the line for which point A is defined and μ_B, μ_C and μ_D have similar interpretation. The cross-ratio is the basic invariant in projective geometry since all other projective invariants can be derived from it. It has been shown [2] that any linear transformation in homogeneous coordinates — like perspective projection, linear scaling, skewing, rotation, translation, etc — preserves this cross-ratio. The cross-ratio of a pencil of four coplanar lines l_1, l_2, l_3 and l_4 going through O , is defined as the cross-ratio $[A, B, C, D]$ of the points of intersection of the four lines with any line l not going through O , and is denoted as $[l_1, l_2, l_3, l_4]$.

Let A, B, C and D be four coplanar points, not three of them co-linear. These points are said to define a projective coordinate system in the plane, $\mathcal{P}$, they belong to. The projective coordinates (k_1, k_2, k_3) of any point P of $\mathcal{P}$ are the three real numbers defined as:

$$\begin{aligned} k_1 &= [CA, CB, CD, CP] \\ k_2 &= [AB, AC, AD, AP] \\ k_3 &= [BC, BA, BD, BP] \end{aligned} \quad (3)$$

Any point on $\mathcal{P}$ is uniquely referenced by its projective coordinates k_1, k_2 and k_3 with respect to the $\{A, B, C, D\}$ projective coordinate system.

Consider, for example point P in figure 2. Given its projective coordinates in the $\{A, B, C, D\}$ projective coordinate system and the Cartesian coordinates of A, B, C and D it is a relatively easy task to determine the Cartesian coordinates of P . First, we recall from equation (3) that k_1 is the cross-ratio of the pencil of lines CA, CB, CD and CP . Let us draw a line l with equation $\vec{b} + \mu\vec{d} = \vec{r}$ which intersects CA, CB, CD and CP at points K, L, J and M respectively. Then according to equation (2)

$$\mu_M = \frac{k_1\mu_K(\mu_J - \mu_L) + \mu_L(\mu_K - \mu_J)}{k_1(\mu_J - \mu_L) + \mu_K - \mu_J} \quad (4)$$

Having obtained μ_M , the Cartesian coordinates of M can be found — by replacing μ by μ_M in the equation of line l — and therefore the equation of the line defined by points C

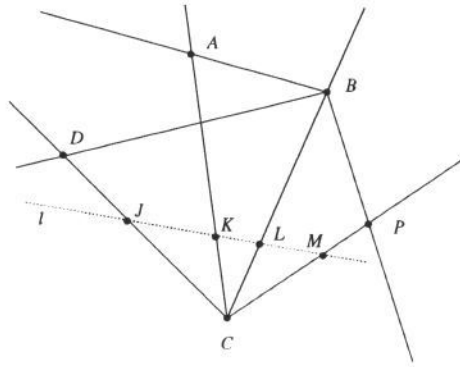


Figure 2: Projective coordinates k_1 , k_2 and k_3 of a point.

and P . In exactly the same way the equations of lines AP and BP can be obtained. The Cartesian coordinates of P are given by the intersection of the three lines AP , BP and CP . It is obvious that to obtain the Cartesian coordinates of P any two lines of AP , BP or CP are enough, thus only two of k_1 , k_2 or k_3 are needed. However, in some cases where degeneration occurs (when point P is collinear with any two reference points) the third value is necessary.

2.2 Projective Reconstruction using planar points

In this section we briefly describe how projective geometry techniques can be used to reconstruct a point P in the world coordinate system, given its left and right image coordinates as well as the exact position and correspondences of a set of eight reference points $\{A, B, C, D, E, F, G, H\}$, consisting of two sets of 4 coplanar points, $\{A, B, C, D\}$ and $\{E, F, G, H\}$.

First, the equation of the viewing line OP from the left camera (figure 3) has to be determined. Consider the first set of reference points $\{A, B, C, D\}$ and their projections to the left image plane $\{a, b, c, d\}$. The projective coordinates k_1, k_2 and k_3 of image point p with respect to the $\{a, b, c, d\}$ projective coordinate system can be determined according to equation (3). If P_1 is the intersection of the viewing line OP with the $ABCD$ plane, the coordinates of P_1 can be determined. The projective coordinates of p in the image plane with respect to $\{a, b, c, d\}$ are the same as the projective coordinates of P_1 in the $ABCD$ plane with respect to $\{A, B, C, D\}$ because of the cross-ratio invariance under perspective projection [4]. Therefore, since the exact positions of A, B, C and D are known, the method described earlier can be used to determine the world coordinates of P_1 . In a similar way the coordinates of point P_2 can be calculated. The two points P_1 and P_2 are enough to define uniquely the equation of the viewing line OP . Working in exactly the same manner the viewing line $O'P$ from the right camera can also be determined. Then, it is trivial to find P as it is the point of intersection of the two viewing lines.

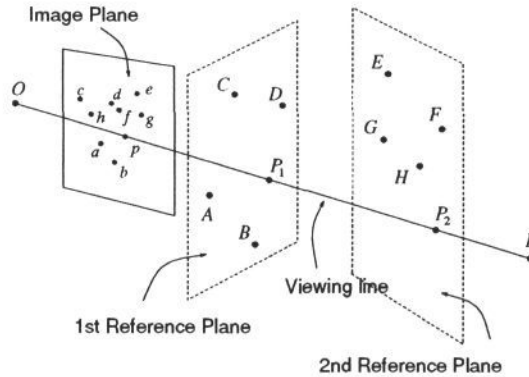


Figure 3: Each of the reference planes contains four reference points. Note that this figure is only schematic and in reality the two sets of reference points will be distinct from each other on the image plane and the quadrilaterals $ABCD$, $EFGH$ will not be one behind the other. The viewing line OP will intersect extensions of the planes on which the reference points lie.

3 Error Analysis

Although the 3D reconstruction technique described in the previous section is accurate and computationally simple, it is very sensitive to noise. The point is that the equations involved are non-linear and thus the propagation of error is not straight forward. Indeed, in non-linear equations it is often the case that the error in the computed quantity is not only a function of the error in the measured quantity times a constant, but it also depends on the computed value itself. Thus, there may be ranges of values for which the error is unacceptably amplified. We shall discuss here the way the error propagates at each stage of the reconstruction process, starting from the estimation of the error in the calculation of the projective coordinates on the image plane, and finishing with the estimation of the error in the calculation of the 3D position of point P .

3.1 Error in the calculation of the projective coordinates

Let us consider the projections of the four reference points on the image plane a , b , c and d and the projection p of the point whose 3D position we want to determine. The first projective coordinate of p with respect to the Cartesian coordinates of a , b , c and d , computed from the pencil of lines with vertex a , can be derived to be:

$$k_2 = \frac{((a_x c_y - a_x p_y + c_x p_y + a_y p_x - a_y c_x - p_x c_y) (a_x b_y - a_x d_y + b_x d_y - d_x b_y - a_y b_x + a_y d_x))}{((a_x c_y - a_x d_y + c_x d_y + a_y d_x - d_x c_y - a_y c_x) (a_x b_y - a_x p_y + b_x p_y + a_y p_x - p_x b_y - a_y b_x))} \quad (5)$$

where (a_x, a_y) , (b_x, b_y) , (c_x, c_y) , (d_x, d_y) , and (p_x, p_y) are the Cartesian coordinates of points a , b , c , d and p respectively. Let us assume that each of the reference pairs of coordinates can be estimated with error normally distributed with zero mean and covariance matrix

$\Sigma = \begin{pmatrix} \sigma_{xx}^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy}^2 \end{pmatrix}$. Then, it can be shown that the variance of the error distribution in the value of k_2 is given by:

$$\begin{aligned} \sigma_{k_2 k_2}^2 = & \left[\left(\frac{\partial k_2}{\partial a_x} \right)^2 + \left(\frac{\partial k_2}{\partial b_x} \right)^2 + \dots \right] \sigma_{xx}^2 + \left[\left(\frac{\partial k_2}{\partial a_y} \right)^2 + \left(\frac{\partial k_2}{\partial b_y} \right)^2 + \dots \right] \sigma_{yy}^2 \\ & + 2 \left[\frac{\partial k_2}{\partial a_x} \frac{\partial k_2}{\partial a_y} + \frac{\partial k_2}{\partial b_x} \frac{\partial k_2}{\partial b_y} + \dots \right] \sigma_{xy} \end{aligned} \quad (6)$$

Applying this formula using k_2 given by (5) we can derive an expression for the error in k_2 which depends on the location of point p on the image plane. If we assume that the errors in x and y are independent from each other and identically distributed, we can set $\sigma_{xx}^2 = \sigma_{yy}^2 = \sigma^2$ and $\sigma_{xy} = 0$ in the above expression. Then, the coefficient which multiplies σ^2 is the error amplification factor. As long, as this factor is less or equal to 1 the error is damped but when this factor exceeds 1, the error is amplified. We can derive similar expressions for the other two projective coordinates of p .

In figure 4 we fixed the positions of the reference points and allowed the position of p to scan the whole plane. We mark with black the regions where amplification of the error is expected. Each black stripe corresponds to error amplification due to one of the projective coordinates. Notice that apart from three small regions around points a , b and d where two projective coordinates are with amplified error, in all other places p has at least two projective coordinates which can be calculated reliably, and this is enough for the determination of the position of point P in the 3D space.

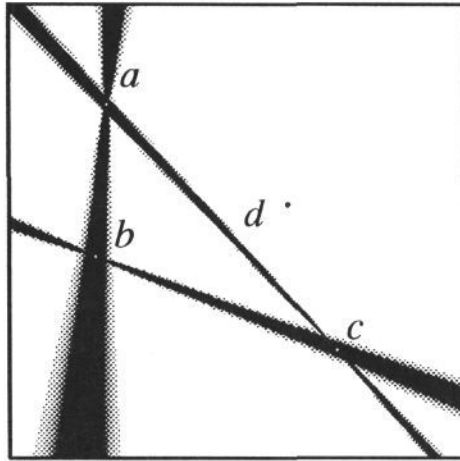


Figure 4: Regions of instability (shown in black).

3.2 From the projective coordinates to the 3D coordinates of P_1

Let us say that of the three projective coordinates of p computed in the previous stage k_1 and k_2 are the most reliable. Point P_1 on the plane defined by points A , B , C and D has the same projective coordinates and the problem now is to find its 3D Cartesian coordinates

from the knowledge of the 3D Cartesian coordinates of A, B, C and D and k_1 and k_2 . Since points A, B, C, D and P_1 belong to the same plane, a translation and rotation transformation can be found from the world coordinate system to a local coordinate system defined in such way that its origin is at A , its y axis is along the line AB and the z axis is normal to the reference plane. The problem is then 2D, and after the straightforward, but tedious, application of simple Geometric and Algebraic reasoning, we can derive the following formulae for the coordinates of P_1 :

$$P_{1x}d = k_1k_2(-A_xB_yC_x + A_xB_yD_x - A_xD_yB_x + A_xD_yC_x + A_xC_yB_x - A_xC_yD_x) + \\ + k_2(-A_xC_yB_x + A_xB_xD_y - C_xD_yB_x + C_yB_xD_x - D_xA_yB_x + C_xA_yB_x) + \\ + (B_xC_xD_y - C_xD_yA_x - C_xB_yD_x - C_xA_yB_x + C_xA_yD_x + A_xB_yC_x) \quad (7)$$

$$P_{1y}d = k_1k_2(+B_yD_xA_y - B_yC_xA_y + C_yB_xA_y - C_yD_xA_y + D_yC_xA_y - B_xA_yD_y) + \\ + k_2(A_xB_yD_y - A_xB_yC_y + B_yC_yD_x - B_yD_yC_x - B_yA_yD_x + B_yC_xA_y) + \\ + (B_xC_yD_y - A_yB_xC_y + C_yD_xA_y - C_yD_xB_y + A_xC_yB_y - C_yA_xD_y) \quad (8)$$

$$\text{with } d = k_1k_2(B_yD_x - C_yD_x - D_yB_x + C_yB_x + D_yC_x - B_yC_x) + \\ + k_2(A_xD_y - C_yA_x - D_yC_x + A_yC_x + C_yD_x - A_yD_x) + \\ + (B_xD_y - A_yB_x - D_xB_y + A_xB_y + A_yD_x - A_xD_y) \quad (9)$$

where all coordinates that appear in these formulae refer to the local coordinate system defined on plane (A, B, C, D) . To investigate the effect of the error in the measured positions of the reference points on the determination of the position of P_1 we can proceed in a way similar to the one described in the previous section. That is, we can derive formulae similar to formula (6) for the error in the calculation of P_{1x} and P_{1y} introduced by the error in the actual positions of the reference points A, B, C and D , assuming that the values of k_1, k_2 and k_3 are known accurately. Then, we repeat the process we followed for the construction of figure 4: As point p scans the image plane we compute at each position the values of k_1, k_2 and k_3 for the given set of reference points.

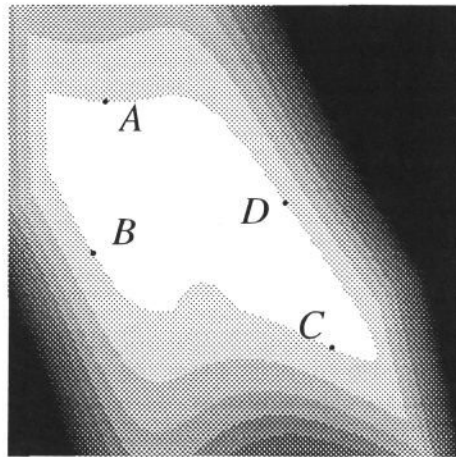


Figure 5: Regions of instability for determination of P_{1x} .

Ignoring the fact that k_1 and k_2 are themselves computed with some error, we put their values into the formulae we derived for the amplification factors and calculate them assuming that the error in all coordinate positions of the reference points is the same. Figure 5 shows the various regions in the image plane where the amplification factor for the error in P_1 is within a certain range. White are the regions where the amplification factor is less than 1, so they are the stable regions. Each shade corresponds to the amplification factor increment by 1 as we move away from the white region, with the very dark regions corresponding to error amplification factor more than 10. Similar analysis can be performed to find the instability regions for P_2 but the only difference with the above analysis is the reference points used.

3.3 Determining the viewing line

Having computed the 3D coordinates of P_1 , the intersection of the viewing line OP with the first reference plane, we can repeat the process and calculate the 3D coordinates of P_2 , the intersection of the viewing line with the second reference plane (see figure 6).

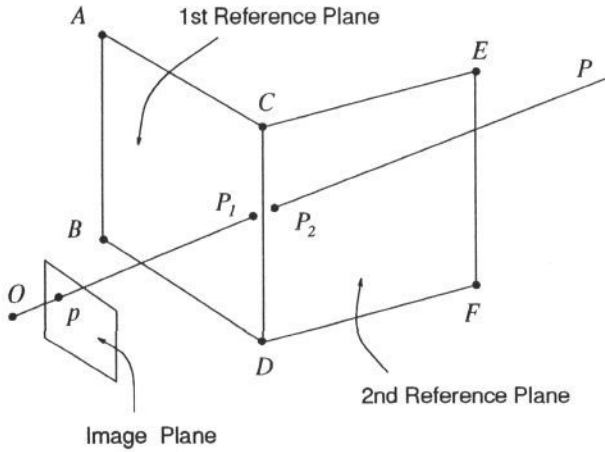


Figure 6: Error is introduced when the distance between the asymptotic lines OP and CD (defined as the minimum distance between any two points, one belonging to one and other to the other line) is small.

Let us say that the position vectors of P_1 and P_2 are $\vec{P}_1$ and $\vec{P}_2$ respectively. Then the position vector of any point P on the viewing line will be given by:

$$\vec{P} = \vec{P}_1 + \mu (\vec{P}_2 - \vec{P}_1) \quad (10)$$

Notice that the parameter μ takes values in the range $[0, 1]$ for points which belong to the segment P_1P_2 and values outside this range for all other points of the line. In terms of coordinates the above equation can be written as:

$$\begin{aligned} P_x &= (1 - \mu)P_{1x} + \mu P_{2x} \\ P_y &= (1 - \mu)P_{1y} + \mu P_{2y} \\ P_z &= (1 - \mu)P_{1z} + \mu P_{2z} \end{aligned} \quad (11)$$

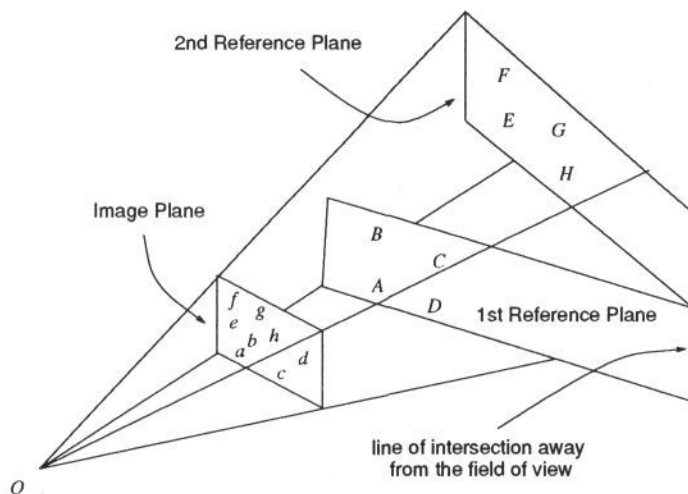


Figure 7: Setup of reference planes and points for minimum sensitivity to noise.

If we consider the coordinates of P_1 and P_2 to be random variables distributed with variance σ_1^2 and σ_2^2 respectively, (for simplicity we assume that we have the same error in both coordinates), the above expressions indicate that the variance of the distribution of the coordinates of P will be $(1 - \mu)^2\sigma_1^2 + \mu^2\sigma_2^2$. This expression shows that only if point P is on the segment P_1P_2 the error is damped. For P in any other position along the viewing line, the error is amplified. The closer points P_1, P_2 are to each other, the greater the amplification factor could be because it is proportional to the square of the distance of P from P_1 measured in units of length $|P_1P_2|$. A good way to avoid this problem is to make sure that the two reference planes of the projective coordinate systems intersect well away from the area of interest. This may not be possible in some cases.

4 Conclusions

In order to apply the method of Projective Geometry to 3D reconstruction we need to take the following cautionary steps to avoid the introduction of large errors:

1. It is best if the two reference planes are apart from each other and intersect along a line well away from the area of viewing. As reference points have to be visible on the image, this requirement implies that a setting like the one shown in figure 7 is appropriate. However, such an arrangement of reference points is not possible in the case of the granite stone reconstruction. We propose instead to use two sets of reference planes; planes $ABCD$ and $AEFB$ for all those points that fall on the right half of the image, and planes $AEFB$ and $CEFD$ for all those points which are on the left half of the image (as shown in figure 6). Such a setting would reduce the source of error amplification described in section 3.3.
2. Provided the accuracy of the measurements of positions on the image plane is known, the error with which each of the projective coordinates of point p is computed can be

estimated and the most reliable coordinates of point p may be used each time. There are only three small patches on the image plane where there is only one reliable projective coordinate.

3. Once the projective coordinates of a point have been found, and given the uncertainty in the measurement of the 3D position of the reference points, the error in the 3D position of point P can be estimated. This stage, however, is the most difficult to handle as it seems that unless the points are projected within a region more or less surrounded by the reference points, we are bound to have amplification of the error. The only thing we can do is to try to monitor it carefully. One can envisage the situation where each side of the granite block is reconstructed with the help of several points. Points which are unreliable then can be dropped out of the process and only points with acceptable accuracy are kept for the reconstruction stage.

Alternatively, one may consider several sets of reference points and compute the position of each point under consideration using all of these sets and every time keep the most reliable set. For the problem of granite block reconstruction, however, this is not easy. Many calibration and reference points are an impractical luxury in a stone processing plant. It seems more practicable to attempt to reconstruct each surface of the stone using several points some of which will have to be discarded during the process.

Acknowledgements

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