Spline Smoothing: A Special Case of Diffusion Smoothing

Li-Dong Cai

Department of Artificial Intelligence, University of Edinburgh
5 Forrest Hill, Edinburgh EH1 2QL, UK

Diffusion Smoothing (DS) implements the smoothing by directly solving a boundary value problem of the diffusion equation $\frac{\partial u}{\partial t} = b\nabla^2 u$ with explicit or implicit numerical schemes, it provides a uniform theoretical base for some other smoothing methods. It has shown that the elegant Gaussian smoothing (GS) is equivalent to the initial value problem of DS, and the widely-used Repeated Averaging (RA) is a special case of the explicit DS. This paper further proves that Spline smoothing (SS) is a special case of the explicit DS with a "convex corner cling" boundary condition. This result coincides with Poggio's conclusion However, our proof starts from the diffusion smoothing theory instead of regularisation theory and is given in the mask form, thus is simpler and more straightforward. Moreover, it makes us possible to explicit the scale space behaviour of spline smoothing.

1. Introduction.

In computer vision, Gaussian Smoothing (GS) is a well-known method due to its elegant properties; whose widely used form is Repeated Averaging (RA) [1,6]. Spline smoothing (SS), initially used to surface approximation, can also be used to smooth surfaces.

Apart from them, [2,3,4,5] discussed a method using the diffusion process to do surface smoothing, or the Diffusion Smoothing (DS) method. This method implements the smoothing by directly solving a boundary value problem of the diffusion equation $\frac{\partial u}{\partial t} = b\nabla^2 u$ with explicit or implicit numerical schemes.

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The DS method has some advantages: 1) It provides a uniform theoretical base for several different smoothing methods, e.g., GC is equivalent to the initial value problem of DS; RA is a special case of the corresponding explicit DS; 2) The implicit DS version works faster in a scale space and produces denser intermediate results without extra computation. 3) It provides an easier and more reasonable way to treat the boundary condition, e.g., shape preservation at the surface margin using a "small leakage" model. 4) It can also be applied to intensity data processing, drifting object smoothing [5] and symmetry axis elicitting [9], etc.

In this paper, we further include SS into DS by showing that SS is a special case of the explicit DS with a "convex corner cling" boundary condition, and discuss the behaviour of SS in scale space.

2. Surface smoothing using cubic B-spline.

Surface spline smoothing method not only gives a smooth version of the surface but also preserves the convexity/concavity of the overall surface shape in the processing [7], thus it seems an attractive smoothing methods.

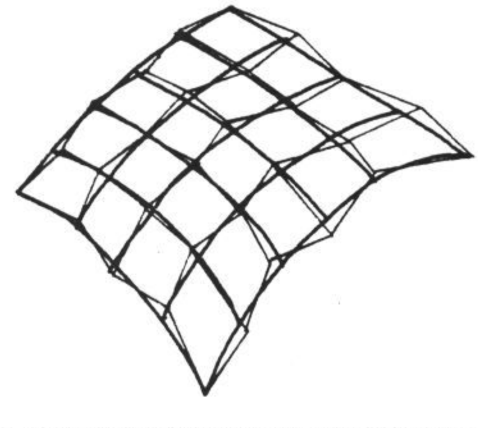

Figure 1. Surfaces from raw date and its spline smoothing approach.

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$\{u_{p,q} \approx u(x_p, y_q) \mid_{p,q=0(1)m-1}\}$ of a surface $u(x,y)>0$ which is discretely sampled on a square mesh π: $[x_p = x_0 + ph, \quad y_q = y_0 + qh \mid_{p,q=0(1)m-1}]$, the intersection of the surface boundary L and the mesh π is a set $L_\pi = \{(s,r) \mid_{0<u_{s,r},\,(s,r)\in L,\,0\le s,r\le m-1}\}$, where m is the mesh width, $h>0$ is the mesh step.

First, along the boundary L_π, we extend the raw data surface $\{u_{p,q} \mid_{p,q=0(1)m-1}\}$ one node outwards from the surface:

$$u_{s-1,r} = 2u_{s,r} - u_{s+1,r} \qquad \text{if } u_{s-1,r} \notin L$$

$$u_{s+1,r} = 2u_{s,r} - u_{s-1,r} \qquad \text{if } u_{s+1,r} \notin L \tag{1.x}$$

$$u_{s,r-1} = 2u_{s,r} - u_{s,r+1} \qquad \text{if } u_{s,r-1} \notin L$$

$$u_{s,r+1} = 2u_{s,r} - u_{s,r-1} \qquad \text{if } u_{s,r+1} \notin L \tag{1.y}$$

Second, holding y constant in u(x,y), we approach the surface u(x,y) in the x-direction and get the spline surface

$$S_x u(x,y) = \sum_{i=-1}^{m} u(x_i, y)\, \phi_i(x) \tag{2.x}$$

and holding x constant in $S_x u(x,y)$, approach the surface $S_x u(x,y)$ in the y-direction to get the final spline surface:

$$S_y S_x u(x,y) = \sum_{j=-1}^{m} S_x u(x, y_j)\, \psi_j(y)$$

$$= \sum_{j=-1}^{m} \sum_{i=-1}^{m} u(x_i, y_j)\, \phi_i(x)\, \psi_j(y) \tag{2.y}$$

Third, replacing $u(x_i, y_j)$ with $u_{i,j}$ gives discrete values of the spline approximation of the surface u(x,y):

$$\hat{u}(x,y) = \sum_{j=-1}^{m} \sum_{i=-1}^{m} u_{i,j}\, \phi_i(x)\, \psi_j(y) \tag{3}$$

Where $\phi_i(x)$ and $\psi_j(y)$ are in the forms of cubic B-spline function $\Omega_3(\cdot)$:

$$\phi_i(x) = \Omega_3(\frac{x-x_i}{h}) = \Omega_3(\frac{x-x_0}{h} - i) \tag{4.x}$$

$$\psi_j(y) = \Omega_3(\frac{y-y_j}{h}) = \Omega_3(\frac{y-y_0}{h} - j) \tag{4.y}$$

So, after spline smoothing, the raw data $\{u_{p,q} \mid_{p,q=0(1)m-1}\}$ will be changed to

$$\hat{u}_{p,q} = \sum_{j=-1}^{m} \sum_{i=-1}^{m} u_{i,j}\, \Omega_3(p-i)\, \Omega_3(q-j) \tag{5}$$

Owing to the properties of the cubic B-spline functions, this approximation has the following characteristics: 1) The surface approximation is of C^2 continuity and $O(h^2)$ accuracy. 2) The surface convexity and concavity are preserved. 3) The computation can be done locally and parallelly within 3×3 windows.

Repeatedly using this spline surface approach, the approximation will converge to a smooth surface with C^2 continuity which preserves the global shape of the surface.

3. Spline Smoothing, Gaussian Smoothing and Repeated Averaging.

Using the mask form, it is easy to prove the result below:

Theorem 1. *Cubic B-spline surface smoothing is a special case of repeated averaging or discrete Gaussian convolution.*

Proof: Due to the compactness of the cubic B-spline functions, at any mesh node $(p-i)$ or $(q-j)$ which is beyond the interval (-2, 2), we have

$$\Omega_3(p-i) \equiv 0 \quad \text{or} \quad \Omega_3(q-j) \equiv 0 \tag{6}$$

Rewrite the spline approach formula (5) as:

$$\hat{u}_{p,q} = \sum_{j=q-1}^{q+1} \sum_{i=p-1}^{p+1} u_{i,j}\, \Omega_3(p-i)\, \Omega_3(q-j) \tag{7}$$

Note that

$$\Omega_3(0) \equiv \tfrac{2}{3} \quad \text{and} \quad \Omega_3(\pm 1) \equiv \tfrac{1}{6} \tag{8}$$

We get a typical Gaussian-like mask form of the surface spline smoothing as below:

$\frac{1}{36} \times$

1	4	1
4	16	4
1	4	1

Figure 2. Surface spline smoothing mask.

So, spline smoothing is a special case of RA. Repeatedly using this mask approximately corresponds to filtering with a Gaussian [4,6].

Q.E.D.

This result coincides with the conclusion in [8] that "the solution to the variational problem $\sum_{k=1}^{n}(f_k - S(x_k))^2 + \lambda \int \| S''(x) \|^2 dx$ in the case of inexact data on a regular grid (and appropriate boundary conditions), can be obtained (a) by convolving the data with a filter, (b) which is a cubic spline, and (c) which is very similar to a Gaussian."

However, by starting from the diffusion smoothing theory instead of regularisation theory, the proof is given in the mask form thus it is in a simpler and more straightforward style. Moreover, it makes us possible to explicit spline smoothing's behaviour in scale space.

4. Spline smoothing and explicit diffusion smoothing.

As well as repeated averaging has been proved as a special case of the explicit diffusion smoothing, we now discuss the relationship between spline smoothing and diffusion smoothing.

Theorem 2. *Surface spline smoothing is a diffusion process with a "convex corner cling" boundary condition:*

$$\frac{\partial u}{\partial t} = \frac{1}{6}\Delta u \qquad (9.e)$$

$$u \Big|_{t=0} = f \qquad (9.i)$$

$$u \Big|_{L_\pi^o} = f \Big|_{L_\pi^o} \qquad (9.b)$$

where f is the input data surface given at the initial time of the diffusion process, L is the surface boundary and L_π^o the "convex corner" node set of L_π whose meaning will be clear in the proof.

Proof: To prove that DS includes SS, we appeal to the general explicit DS scheme and its mask. This scheme is a linear combination of the normal and oblique cross explicit schemes in [2]:

$$u_{i,j}^{k+1} = \omega[(1-4\beta)u_{i,j}^k + \beta(u_{i-1,j}^k + u_{i+1,j}^k + u_{i,j-1}^k + u_{i,j+1}^k)]$$

$$+(1-\omega)[(1-2\beta)u_{i,j}^k + \frac{\beta}{2}(u_{i-1,j-1}^k + u_{i+1,j+1}^k + u_{i-1,j+1}^k + u_{i+1,j-1}^k)]$$

(10)

$\frac{(1-\omega)\beta}{2}$	$\omega\beta$	$\frac{(1-\omega)\beta}{2}$
$\omega\beta$	$1-2(1+\omega)\beta$	$\omega\beta$
$\frac{(1-\omega)\beta}{2}$	$\omega\beta$	$\frac{(1-\omega)\beta}{2}$

Figure 3. The general explicit DS mask.

Where $0 \le \omega \le 1$ is the weight coefficient, $i, j = 0\ (1)\ m-1$ in the spatial step h, $k = 0\ (1)\ \infty$ in the diffusion time step τ, $b > 0$ the diffusion coefficient and $\beta \equiv \frac{b\tau}{h^2}$.

Comparing this general DES mask with the spline smoothing mask in Figure 2 gives $\omega = \frac{2}{3}$ and $\beta = \frac{1}{6}$. Setting $\tau = h = 1$ yields the diffusion coefficient $b = \frac{1}{6}$. So spline smoothing is a special case of the explicit diffusion smoothing.

Because the boundary treatment is performed with a linear extension formula (1), surface values at some points are invariant during the smoothing process. These points are just the convex corner nodes of the L_π shown in Figure 4 as bold dots and are grouped into a set $L_\pi^o = \{(s,r) \mid_{(s,r) \in L,\ 0 < u_{s,r}^k = u_{s,r}^o,\ 0 \le s,r \le m-1,\ 0 \le k < \infty}\}$.

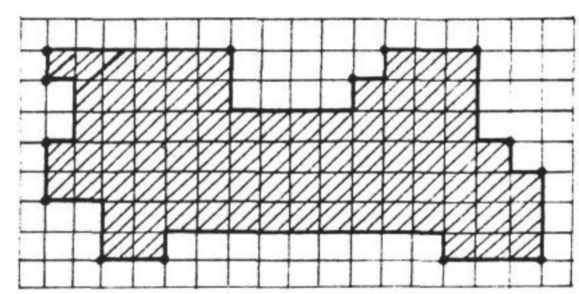

Figure 4. Convex corner node set L_π^o (blackened dot nodes of L_{pi}) in spline smoothing.

This means that under such a boundary treatment, the spline surface must cling to the input data surface on the L_π^o throughout the diffusion process. In return, we name it a treatment with the "convex corner cling" boundary condition formulated as in (9).

Q.E.D.

Corollary. *Repeatedly using the surface spline approach n times approximately corresponds to filtering with a Gaussian whose standard deviation is* $\sqrt{\frac{n}{3}}$.

Proof: Because SS is a special case of the diffusion process presented as in (9), by setting $t = n$ and $b = \frac{1}{6}$ in the following relationship (11) between Gaussian scale σ and diffusion time t, the conclusion follows.

$$\sigma = \sqrt{2bt} \tag{11}$$

Q.E.D.

5. Scale Space Behaviour of Spline Smoothing.

We compare the scale space behaviour at the surface boundary and computational performance of spline smoothing with that of implicit diffusion smoothing. First, the cubic B-spline smoothing is a special case of explicit diffusion smoothing. Due to the constraint of numerical stability, it promises a much lower computational efficiency than the implicit diffusion smoothing in the scaled space (cf. [4]).

Second, the surface boundary treatment in spline smoothing leads to the smoothed version cling to the raw data surface at those convex corner nodes at the boundary. In the case of once approaching to the noise-free data, this invariance would not be a problem, even an advantage; but in the case of repeatedly smoothing the noisy data, it might provide false information about the surface tendency at the boundary and cannot be corrected throughout the whole process, whereas the surface boundary treatment is better in the diffusion smoothing with a "small leakage model" where surface curvature signs are preserved at the surface boundary [5].

Third, the weight coefficient $\omega = \frac{2}{3}$ in the proof of theorem 2 suggests spline smoothing mask is not an isotropic filter while many other diffusion smoothing masks can be isotropic ones.

Section 6. Summary.

In this paper, spline smoothing has been proved as the diffusion process with a "convex corner cling" boundary condition, which explicits spline smoothing's relationships to Gaussian smoothing, repeated averaging and diffusion smoothing;then the scale space behaviour of spline smoothing is compared with that of implicit diffusion smoothing.

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