AVC89: A Method for Retrieving Images from Noisy Incomplete Data

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We propose here a new method for retrieving an image from noisy, incomplete data. It is based on an idea incorporating the principle of maximum entropy, a well known powerful method for image processing. An image is built on the computer by allocating grains to pixels one at a time on the basis of certain deterministic rules as to where each successive grain is placed in image space. The performance of the method is tested in an application in astronomy and for two-dimensional model images.

We propose here a simple and effective method for reconstructing an image from noisy, incomplete data. The essence of the method is to allocate trial grains one at a time to an image space in which an image is constructed by a set of deterministic rules for the placement of each successive grain. It is therefore called the grain allocation method (GRALM). Frieden [1] was the first to use this idea in processing strongly blurred alphanumeric characters.

In this paper, the idea is further developed and combined with the principle of maximum entropy [2], which makes it possible to construct, in the class of all feasible images (i.e., those compatible with the given data), the one which is most unbiased. Reconstructions with greater entropy represent more disorder, and are more probable and more natural according to Shannon's interpretation of entropy [3] as a measure of information.

GRAIN ALLOCATION METHOD (GRALM)

Let the required image have pixel values represented by positive numbers $f_1, f_2, \ldots, f_N$, which are to be determined, and the observed data be given by

$$d_i = \sum_{j=1}^{N} R_{ij} f_j + n_i, \quad i = 1, \ldots, M, \qquad (1)$$

where the n_i represent random noise and the set (R_{ij}) is the point-spread function.

Let us assume that each f_j consists of a number of grains, each of intensity Δf, and the image space in which the f_j are defined is formed by dividing it into N pixels ($j=1,\ldots,N$). Grains are then allocated one at a time to initially empty pixels on the basis of certain decision rules. Since the image reconstruction problem is ill-posed (i.e., small perturbations in the data may give large errors in the reconstruction), such problems generally require some kind of regularization in order to generate physically plausible reconstructions. Practical methods of regularization are based on building smoothness into the reconstruction. This can be done in a variety of ways, but the one which is more effective is the maximum entropy approach [4] that results in the most featureless model consistent with the data.

Different versions of the maximum entropy approach have been used by different authors, but the numerical problem [5] is in general one of a constrained optimization. Our formulation is to maximize the functional S, called the entropy, defined as

$$S = -\sum_{j=1}^{N} P_j \log P_j , \qquad (2)$$

where $P_j = f_j/\Sigma f$, subject to a constraint of the form

$$Q(f) \le Q_{max} \qquad (3)$$

which expresses the requirement of statistical consistency with the actual data. The statistic Q(f) is usually chosen as the chi-squared value [6], defined in the form

$$Q(f) = \sum_{i=1}^{M} \frac{1}{\sigma_i^2} \left[d_i - \left(\sum_{j=1}^{N} R_{ij} f_j \right) \right]^2 , \qquad (4)$$

where σ_i is the standard deviation of the noise in the data, with the constraint $Q(f) \le M$. Our maximum entropy procedure requires an initial feasible reconstruction, i.e., one which satisfies this consistency requirement. For this purpose, an optimization problem whose cost function is related to how well the constraint (3) is satisfied is modelled, the cost function being given by

$$Q_k(f) = \sum_{i=1}^{M} \frac{1}{\sigma_i^2} \left[d_i - \gamma^k d_i^k \right]^2 . \qquad (5)$$

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The d_i^k form the cumulative image after placing the k^{th} grain in the pixel J, and are thus given by

$$d_i^k = d_i^{k-1} + R_{iJ}\,\Delta f. \tag{6}$$

γ_k^k is a normalization parameter, defined as $\gamma^k = (\sum f)/k$. To place each grain, we seek the pixel which gives the largest reduction in Q(f); we find the pixel number J which gives the minimum

$$\min_{1\le J\le N}\left[Q_k\left(f_1^{k-1}, f_2^{k-1}, \ldots, f_J^{k-1} + \Delta f, \ldots, f_N^{k-1}\right)\right]. \tag{7}$$

Then the corresponding reconstructed image will be

$$f_J^k = f_J^{k-1} + \Delta f. \tag{8}$$

As the f_j^k are incremented grain by grain, the algorithm at some stage may be trapped in a local minimum because of the definition of a grain as a finite increment. It may then be impossible for a single grain to produce the jump needed to reduce the cost function $Q_k(f)$. So an algorithm that allows $Q_k(f)$ to increase at some stage in order to escape the local minimum is used. Grains are allocated to pixels until a feasible reconstruction which satisfies the consistency condition is obtained. However, many feasible solutions can be found, and the maximum entropy criterion is used to select a particular one. We continue adding grains one at a time to pixels in such a way that the pixel which gives the cumulative image with the greatest entropy is chosen at each stage from among those that satisfy the consistency condition. In other words, a grain is added to the pixel which satisfies the consistency condition and gives largest entropy,

$$\max_{1\le J\le N}\left[S\left(f_1^{k-1}, f_2^{k-1}, \ldots, f_J^{k-1} + \Delta f, \ldots, f_N^{k-1}\right)\right]. \tag{9}$$

This iterative procedure is stopped when

$$\left|\frac{S(f^k) - S(f^{k-1})}{S(f^k)}\right| \le \varepsilon, \tag{10}$$

where ε is preassigned number.

COMPUTER SIMULATIONS

We tested the method described in the previous section for one-dimensional astronomical images and two-dimensional model images. The results are given in this section.

The astronomical data, provided by the School of Physics and Space Research at the University of Birmingham, are a software simulation of the response of the FOURPI instrument, an x-ray all-sky monitor proposed as a possible instrument for the spectrum-X mission (G. K. Skinner, private communication). This consists of four modules, each containing 30 sectors. Each module can be thought of as a number of conventional coded mask telescopes in each of which the mask pattern is one-dimensional; the mask is smaller than the detector in one dimension, and both the detector and the mask pattern are wrapped around the circumference of a cylinder.

The data are observed as a function of position along the detector of each sector and show a characteristic shadow of the mask pattern, as illustrated in figures 1-5. The

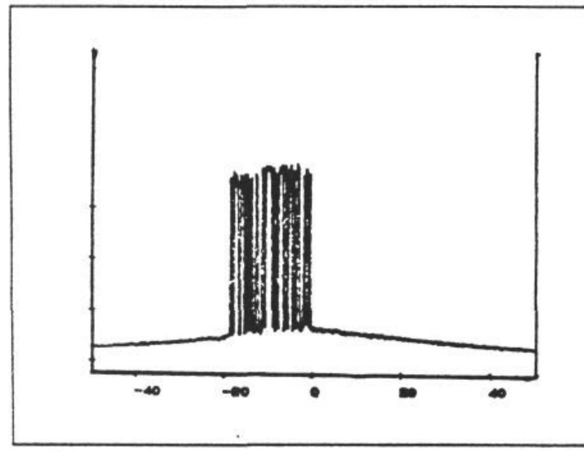

Figure 1. Data observed by the FOURPI instrument.

horizontal axis measures the distance along the detector, expressed as an equivalent angle in degrees. The vertical axis measures the intensity in arbitrary units (in the actual calculations, measured as the number of counts). The noise in the data was generated using a Poisson distribution.

In figure 1, the sector illustrates views of the region of GX17+2, modelled with intensity of 950 Uhuru flux units (ufu). The

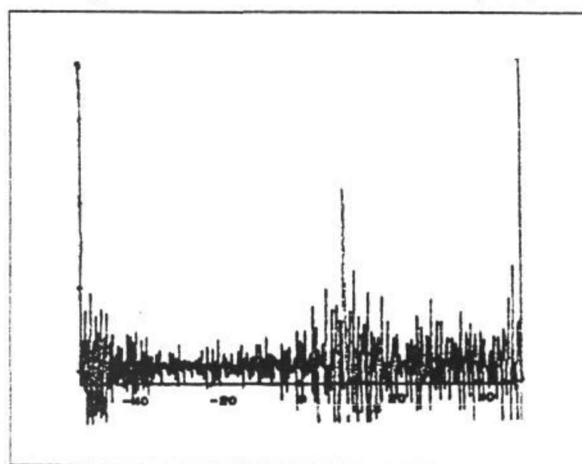

Figure 1.1 Reconstructed image obtained by the direct matrix inversion method.

data were obtained from a simulation of a 8×10^4 s observation. In figure 1.1, the reconstructed image obtained by direct matrix inversion of (1) exhibits large oscillations, and the x-ray source is almost unnoticeable. In figure 1.2, the reconstructed image obtained by the GRALM reveals clearly the x-ray source GX17+2 in the right position with almost the correct intensity. In figures 2-5 the data were obtained from a simulation of a 10^4 s observation. The images retrieved from those data are given in figures 2.1, 3.1, 4.1 and 5.1. Parameters characterizing the computer simulations of the GRALM for the five cases which correspond to figures 1-5 are given in table 1. Here K is the number of the final iteration in the maximum

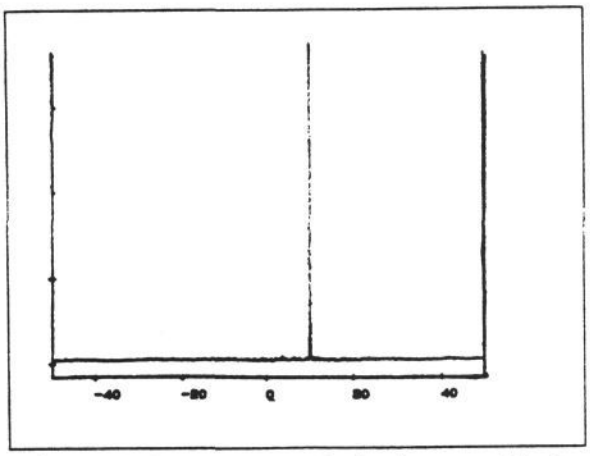

Figure 1.2 Reconstructed image obtained by the GRALM.

	1	2	3	4	5
σ	26.2	4.5	3.15	3.7	10.17
STN	239	35	16	22	89
Q1	398.9	398.5	398.9	398.4	399.61
Q2	399.99	399.96	399.90	399.94	399.62
M	400	400	400	400	400
K	1001	1001	1001	1001	1
S	3.82	3.73	5.21	4.43	-----
ε	0.001	0.001	0.001	0.001	0.001
CPU1	2.5	0.40	0.54	0.51	0.4
CPU2	17.4	15.4	15.5	15.5	0.6
Max(f)	192000	474	394	1644	33098
Max(f)	200000	2690	562	1720	33124

Table 1. Computer simulations of the GRALM

entropy procedure, CPU1 is the Multics CPU time (min) for the feasible solution, CPU2 is the total Multics CPU time (min) for the complete reconstruction, Q1 is the chi-squared value for this feasible solution, Q2 is the chi-squared value for the maximum entropy solution, and STN is the signal-to-noise ratio, defined as STN = $\max(d_i)/\sigma$, i=1,...,M.

Overall, the results indicate very good perfomance in the reconstruction of blurred, noisy images. However, the perfomance in the reconstruction of extremely noisy images

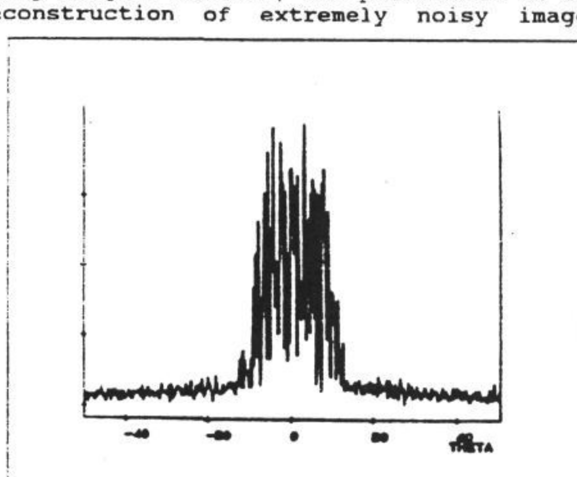

Figure 2. Simulation of data obtained by the FOURPI.

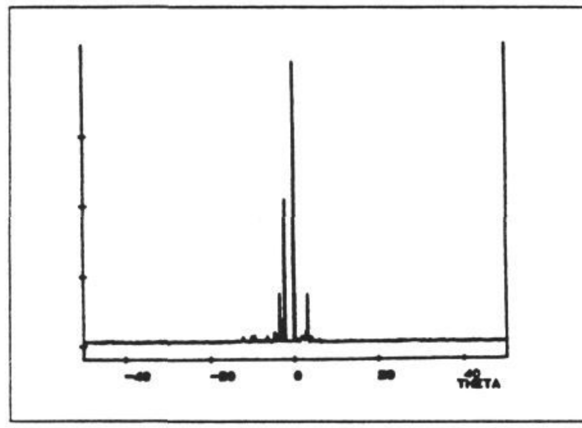

Figure 2.1 Reconstructed image obtained by the GRALM.

seems to be a little poorer. In figures 1.2 and 5.1, the signal-to-noise ratio (STN) is high. Therefore, the maxima of the reconstructed sources in table 1 [Max(f*)] are found with almost the same intensity as the maxima of the original sources taken from the x-ray catalogue [Max(f)]. However, with decreasing STN in table 1, a decrease

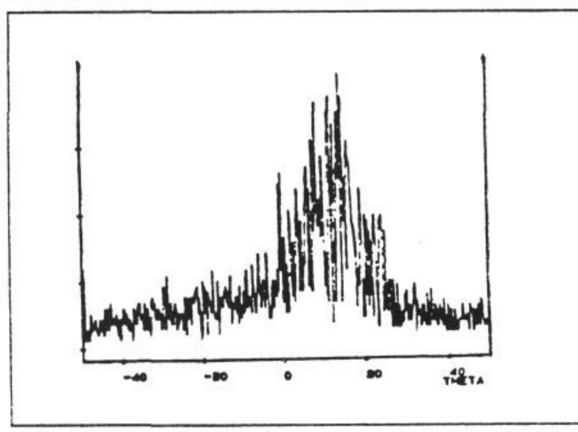

Figure 3. Simulation of data obtained by the FOURPI.

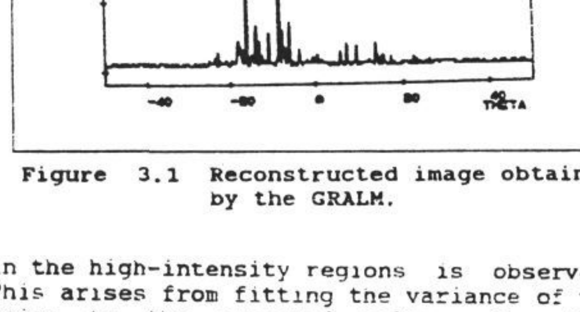

Figure 3.1 Reconstructed image obtained by the GRALM.

in the high-intensity regions is observed. This arises from fitting the variance of the noise to the expected value. In this connection, spurious oscillations due to the noise in the data can be seen in figures 2.1, 3.1 and 4.1. As can be seen in figures 1.1-4.1, the reconstructions contain a uniform background intensity because the maximum entropy criterion favours a uniform distribution. This backgound intensity level varies with the standard deviation of the

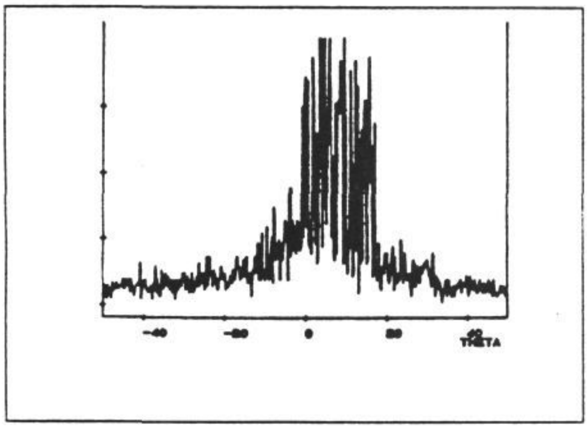

Figure 4. Simulation of data obtained by the FOURPI.

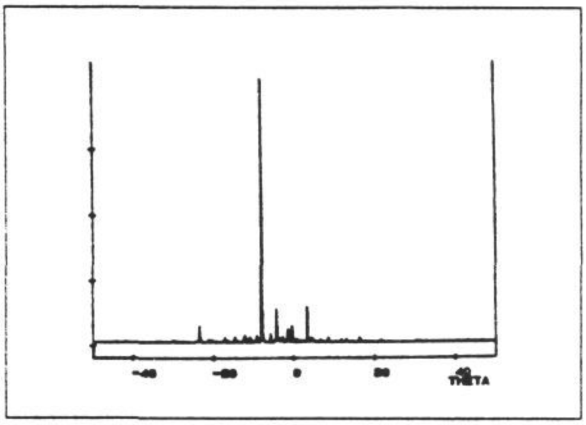

Figure 4.1 Reconstructed image obtained by the GRALM.

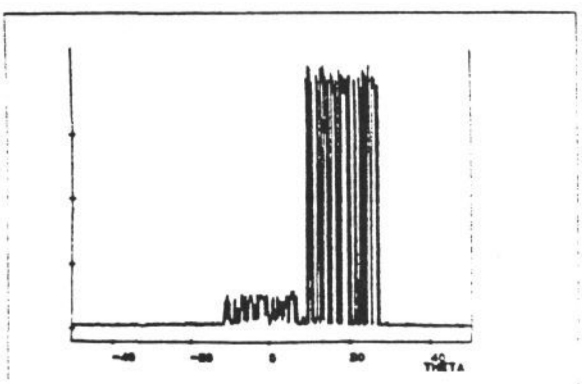

Figure 5. Simulation of data obtained by the FOURPI.

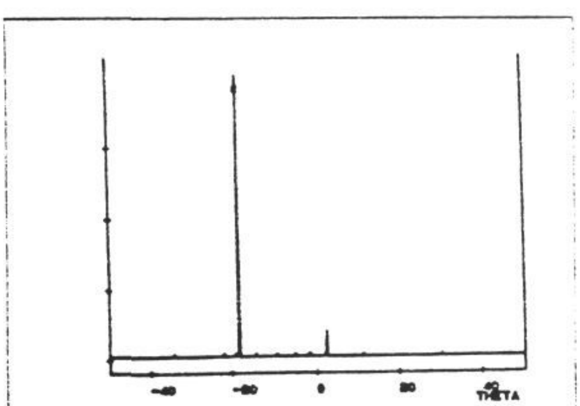

Figure 5.1 Reconstructed image obtained by the GRALM.

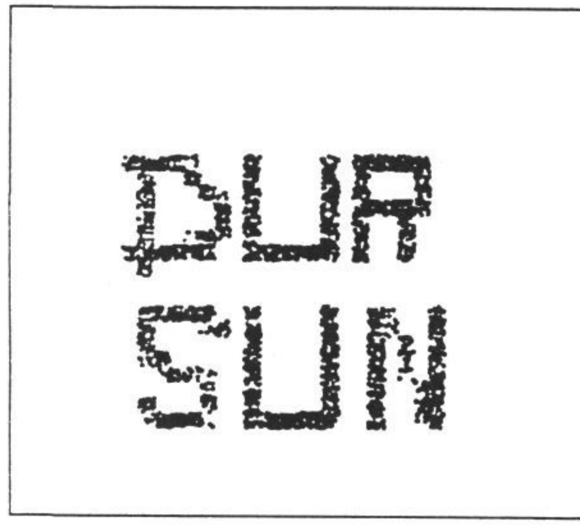

Figure 6. Original image.

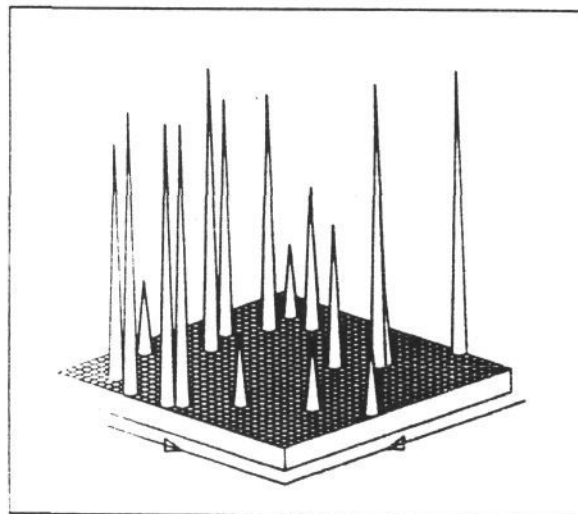

Figure 7. Original image.

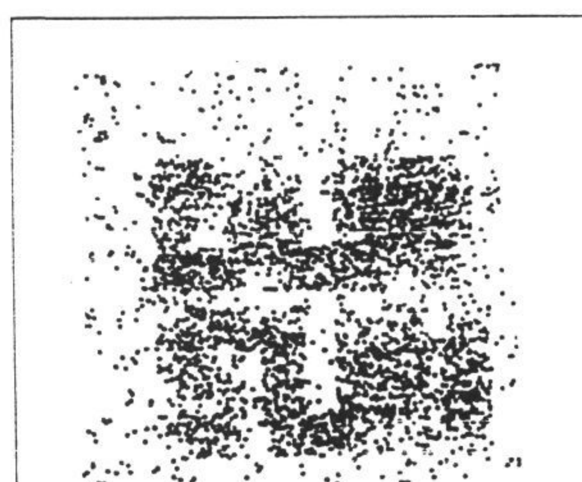

Figure 6.1 Noisy, degraded image.

noise in the data. Consequently, the weak sources whose intensities are less than the standard deviation of the noise are removed. In table 1, it can be seen that the CPU time taken for each simulation depends on the background noise level in the data, and most of the time is spent in stabilizing the solution in the maximum entropy procedure.

The one-dimensional problem discussed above is extended to two-dimensional problems by generating the two-dimensional model images

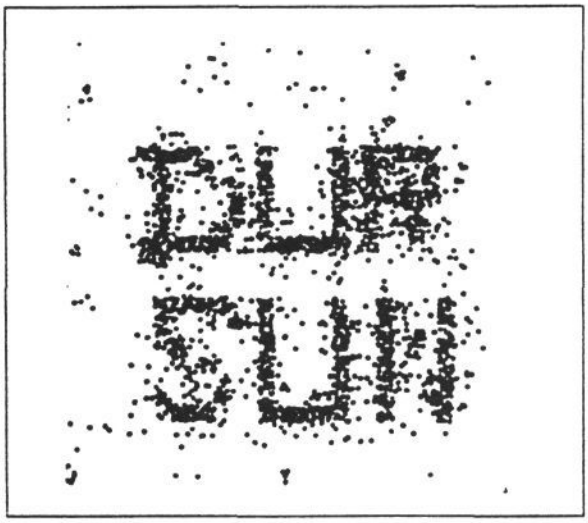

Figure 6.2 Reconstructed image obtained by the GRALM.

shown in figures 6 and 7 with sizes of 30×30 and 32×32. The model images are degraded by a simple point-spread function which averages the intensities over neighbouring pixels (a 3×3 box filter). The noisy data shown in figures 6.1 and 7.1 are obtained by adding random noise to the degraded image. The random noise was generated from the zero-mean Gaussian normal distribution with standard deviation of unity. Figures 6.2 and 7.2 show corresponding reconstructions

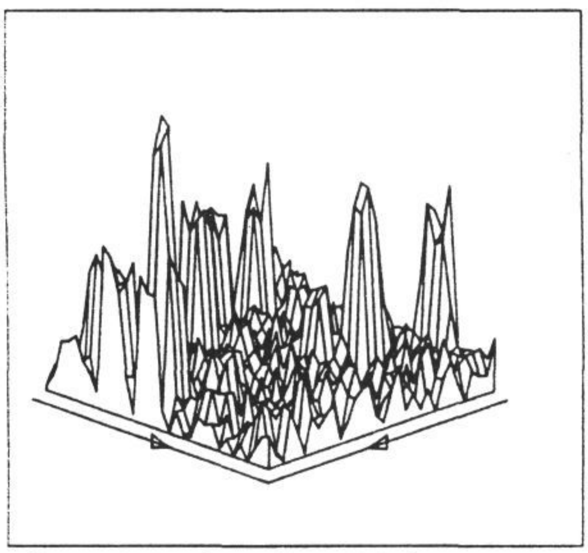

Figure 7.1 Noisy, degraded image.

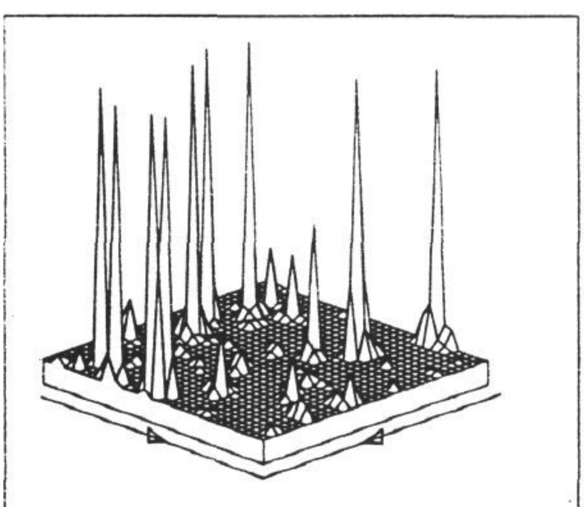

Figure 7.2 Reconstructed image obtained by the GRALM.

obtained by the use of the GRALM. In figure 6.1, the letters R an N are not recognizable, but in figure 6.2 they are clearly recovered. Boundaries of the letters are clearly detected. In figure 7, a three-dimensional representation of an image which consists of randomly placed point sources is shown. Although the two point sources in figure 7 are degraded to the one point source shown in figure 7.1, they are recovered in figure 7.2 in the right positions with an error of a few percent in the intensity.

CONCLUSION

We have described a new method for retrieving images from noisy, incomplete data which incorporates the principle of maximum entropy. The results presented here indicate that the method is a valuable prospective tool for images of impulse and edge types, and gives greatly improved resolution, free from ringing. As expected, the quality of the reconstruction is better for lower noise levels.

We note that much of the computation is very suitable for implementation on parallel computers. A parallel implementation of the algorithm will therefore be considered. An extension of the one-dimensional astronomical imaging problem introduced here to two-dimensional cases in which the mask has the same size as the detector will be considered in a future study.

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