AVC89: Rigid Velocities Compatible with Five Image Velocity Vectors

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The problem of obtaining rigid velocities compatible with a given set of image velocity vectors is algebraic in that it depends on the solution of simultaneous polynomial equations. We show that five image velocity vectors yield two quartic polynomial constraints on the translational part of the rigid velocity, and that of the 16 common zeros of these two quartics, exactly ten yield rigid velocities compatible with the image velocities. An alternative argument that there are in general exactly ten rigid velocities compatible with five given image velocities is briefly sketched.

The fact that as many as ten rigid velocities are obtained indicates that the problem of finding rigid velocities compatible with image velocities is intrinsically difficult.

A body moving relative to a camera gives rise to an image that changes over time. These image changes are described by a 2D motion field of velocity vectors defined on the projection surface of the camera [1, 2, 3]. Information about the motion and shape of the body can be obtained from the 2D motion field. In particular, if the 2D motion field arises from a single rigid moving body, then it can yield the shape and velocity of the body up to a single unknown scale factor [1, 2, 3, 4].

The problem of recovering the shape and velocity of a rigid body from the associated 2D motion field is algebraic, because it depends on solving a number of simultaneous polynomial equations. We investigate the properties of these equations, and show that there are, in general, exactly ten rigid velocities compatible with a given 2D motion field containing five image velocity vectors. The figure ten in this context is a fundamental measure of the complexity of the problem of recovering rigid body motion from image velocities, analogous to the degree of an algebraic curve. Ten is considered high, indicating that the problem is difficult.

The phrase 'in general' means that although some 2D motion fields containing five image velocity vectors are not compatible with exactly ten rigid velocities, such 2D motion fields form a negligibly small part of the space of all 2D motion fields containing five image velocity vectors.

We describe some general properties of polynomials and then obtain the equations for the 2D motion field arising from a single moving rigid body. These equations yield two quartic polynomial constraints, $q_1(\mathbf{v}) = q_2(\mathbf{v}) = 0$, on the translational velocity, $\mathbf{v}$, of the rigid body. We show that $q_1(\mathbf{v})$ and $q_2(\mathbf{v})$ have, in general, exactly 16 distinct common zeros, and we show that, in general, ten of these common zeros yield values of $\mathbf{v}$ compatible with the 2D motion field. An alternative proof that there are ten values of $\mathbf{v}$, based on the theory of ambiguous surfaces, is sketched. Finally some questions are raised concerning the connections between the theory of image velocities and the theory of image displacements.

POLYNOMIALS

We use $\mathcal{R}^n$ to denote n-dimensional Euclidean space, and $\mathcal{P}^n$ to denote n-dimensional projective space. The points of $\mathcal{P}^n$ are represented by $n+1$-tuples of coordinates such that at least one coordinate is non-zero. Two points $\mathbf{x}$, $\mathbf{y}$ of $\mathcal{P}^n$ with coordinates x_i, y_i are identified if and only if there exists a non-zero scalar λ such that $x_i = \lambda y_i$ for $1 \le i \le n+1$.

Let $\mathbf{x} = (x_1, x_2, x_3)$. A polynomial $f(\mathbf{x})$, homogeneous in the coordinates of $\mathbf{x}$, defines a plane curve in $\mathcal{P}^2$. A point $\mathbf{u}$ is on $f(\mathbf{x})$ if and only if $f(\mathbf{u}) = 0$. Let $\mathbf{h} = \nabla f(\mathbf{x})|_u \equiv (\partial f/\partial x_1,\ \partial f/\partial x_2,\ \partial f/\partial x_3)$ evaluated at $\mathbf{u}$. A point $\mathbf{y}$ is on the tangent line to $f(\mathbf{x})$ at $\mathbf{u}$ if and only if $\mathbf{h}.\mathbf{y} = 0$. If $\mathbf{h} = 0$ then $f(\mathbf{x})$ is said to have a singular point at $\mathbf{u}$. The condition that $f(\mathbf{x})$ has a singular point somewhere in $\mathcal{P}^2$ is expressible as a polynomial constraint on the coefficients of $f(\mathbf{x})$.

Plane curves $f(\mathbf{x})$, $g(\mathbf{x})$ are said to intersect transversly at $\mathbf{u}$ if $f(\mathbf{u}) = g(\mathbf{u}) = 0$ and $\nabla f(\mathbf{u}) \times \nabla g(\mathbf{u}) \ne 0$. Transverse intersections are stable, in that if $f(\mathbf{x})$, $g(\mathbf{x})$ are subject to small perturbations then the perturbed polynomials intersect transversely at a point near to $\mathbf{u}$.

Plane curves $f(\mathbf{x})$, $g(\mathbf{x})$ of degrees m, n respectively, intersect at mn points, and these points are distinct if and only if each intersection is transverse. If $f(\mathbf{u}) = g(\mathbf{u}) = 0$, but $f(\mathbf{x})$, $g(\mathbf{x})$ do not intersect transversely at $\mathbf{u}$ then $f(\mathbf{x})$, $g(\mathbf{x})$ are said to have a multiple common zero at $\mathbf{u}$.

A property defined on $\mathcal{P}^n$ holds in general if it holds on an open dense set of $\mathcal{P}^n$. If a polynomial is non-zero at just one point of $\mathcal{P}^n$ then it is non-zero on an open dense set of $\mathcal{P}^n$, thus a polynomial defined on $\mathcal{P}^n$ is either identically zero or it is in general non-zero.

Further information on polynomials, curves and pro-

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jective spaces is given in [5].

2D MOTION FIELDS

The 2D motion field is defined to be the projection of the three dimensional velocity of a body onto the projection surface of the camera [1,2]. The velocity of a point on the body surface projects to an image velocity vector, $\dot{\mathbf{Q}}_i$, and the point itself projects to the base point, $\mathbf{Q}_i$, of $\dot{\mathbf{Q}}_i$. We assume that the body is rigid, and we assume that the image is formed by polar projection onto the unit sphere, centred at the projection point.

The velocity of a rigid body in space is described by a translational velocity $\mathbf{v}$, and an angular velocity $\boldsymbol{\Omega}$. (See [6]). The axis of $\boldsymbol{\Omega}$ is chosen to pass through the centre of the projection sphere. With this choice of axis, $\mathbf{v}$, $\boldsymbol{\Omega}$ are uniquely determined by the motion of the rigid body.

The main equations

The image velocity vectors, $\dot{\mathbf{Q}}_i$, with base points, $\mathbf{Q}_i$, are related to $\mathbf{v}, \boldsymbol{\Omega}$ as follows [1,2,3,4]

$$\dot{\mathbf{Q}}_i = [\mathbf{v} - (\mathbf{v}.\mathbf{Q}_i)\mathbf{Q}_i]K_i + \boldsymbol{\Omega} \times \mathbf{Q}_i \qquad (1)$$

where K_i is the inverse distance to the rigid body surface in the direction $\mathbf{Q}_i$. A value of $\mathbf{v}$ is said to be compatible with the 2D motion field if there exist associated values of $\boldsymbol{\Omega}$ and K_i such that (1) holds.

It follows from (1) that $\mathbf{v} = 0$ is, in general, not compatible with the 2D motion field because the $\dot{\mathbf{Q}}_i$ would otherwise be coplanar. As we only consider the general case, we assume $\mathbf{v} \neq 0$. If $\mathbf{v} \neq 0$ is compatible with (1) then $\lambda\mathbf{v}$ is compatible with (1) for any $\lambda \neq 0$. It is thus natural to regard $\mathbf{v}$ as an element of $\mathcal{P}^2$.

Define $\mathbf{R}_i$ by $\mathbf{R}_i \equiv \dot{\mathbf{Q}}_i \times \mathbf{Q}_i$. On taking the dot product of (1) with $\mathbf{Q}_i \times \mathbf{v}$, we eliminate K_i to obtain

$$\mathbf{R}_i.\mathbf{v} = [(\mathbf{v}.\mathbf{Q}_i)\mathbf{Q}_i - \mathbf{v}].\boldsymbol{\Omega} \qquad (2)$$

The $\mathbf{R}_i$, $\mathbf{Q}_i$ are known quantities obtainable from the image, and $\mathbf{v}$, $\boldsymbol{\Omega}$ are unknown quantities, to be determined by solving (2). Our main result is that if five pairs $\dot{\mathbf{Q}}_i$, $\mathbf{Q}_i$ are given then there are exactly ten values of $\mathbf{v}$ satisfy (2), and hence satisfy (1). We assume from now on that $1 \leq i \leq 5$.

No solutions are lost

We show that we do not overlook any solutions to (1) by passing to (2). In other words, we show that if $\mathbf{v} \neq 0$ is compatible with (2) then $\boldsymbol{\Omega}$, K_i can be found such that (1) holds. We obtain from (2)

$$(\dot{\mathbf{Q}}_i - \boldsymbol{\Omega} \times \mathbf{Q}_i).(\mathbf{v} \times \mathbf{Q}_i) = 0$$

Let $\mathbf{v} \times \mathbf{Q}_i \neq 0$ for all i. Then there exist scalars a_i, b_i such that

$$\dot{\mathbf{Q}}_i = a_i\mathbf{v} + b_i\mathbf{Q}_i + \boldsymbol{\Omega} \times \mathbf{Q}_i \qquad (3)$$

On taking the scalar product of (3) with $\mathbf{Q}_i$ we obtain $a_i\mathbf{v}.\mathbf{Q}_i + b_i = 0$, thus

$$\dot{\mathbf{Q}}_i = [\mathbf{v} - (\mathbf{v}.\mathbf{Q}_i)\mathbf{Q}_i]a_i + \boldsymbol{\Omega} \times \mathbf{Q}_i \qquad (4)$$

Equation (1) follows from (4) on setting $K_i = a_i$.

We complete the proof by showing that $\mathbf{v} \times \mathbf{Q}_i \neq 0$ for all i. If, for example, $\mathbf{v} \times \mathbf{Q}_5 = 0$, then without loss of generality, $\mathbf{v} = \mathbf{Q}_5$. We write the first four equations of (2) in the matrix form $\mathbf{r}(\mathbf{v}) = M(\mathbf{v})\boldsymbol{\Omega}$. As $\boldsymbol{\Omega}$ varies over $\mathcal{R}^3$, $M(\mathbf{Q}_5)\boldsymbol{\Omega}$ varies over a subspace $\mathcal{S}$ of $\mathcal{R}^4$ of dimension at most three. If (2) has a solution with $\mathbf{v} = \mathbf{Q}_5$ then $\mathcal{S}$ includes $\mathbf{r}(\mathbf{Q}_5)$, however, $\mathbf{r}(\mathbf{Q}_5)$ varies over the whole of $\mathcal{R}^4$ as the $\mathbf{R}_i$ vary, thus $\mathbf{r}(\mathbf{Q}_5) = M(\mathbf{Q}_5)\boldsymbol{\Omega}$ does not, in general, have a solution for $\boldsymbol{\Omega}$.

Properties of $(\mathbf{v}.\mathbf{Q}_i)\mathbf{Q}_i - \mathbf{v}$

We require the following two results: if $\mathbf{v}$ varies, with the $\mathbf{Q}_i$ fixed and in general position then (i) no three of the vectors $(\mathbf{v}.\mathbf{Q}_i)\mathbf{Q}_i - \mathbf{v}$ are ever collinear; and (ii) the five vectors $(\mathbf{v}.\mathbf{Q}_i)\mathbf{Q}_i - \mathbf{v}$ always span $\mathcal{R}^3$. The proofs are omitted.

Define cubic polynomials $f_{ijk}(\mathbf{v})$ by

$$f_{ijk}(\mathbf{v}) \equiv \det \begin{pmatrix} (\mathbf{v}.\mathbf{Q}_i)\mathbf{Q}_i - \mathbf{v} \\ (\mathbf{v}.\mathbf{Q}_j)\mathbf{Q}_j - \mathbf{v} \\ (\mathbf{v}.\mathbf{Q}_k)\mathbf{Q}_k - \mathbf{v} \end{pmatrix} \qquad (5)$$

where $1 \leq i < j < k \leq 5$. We show that the $f_{ijk}(\mathbf{v})$ do not, in general, possess a singular point.

The $f_{ijk}(\mathbf{v})$ form a family of polynomials indexed by the components of $\mathbf{Q}_i, \mathbf{Q}_j, \mathbf{Q}_k$. The condition that $f_{ijk}(\mathbf{v})$ possess a singular point can be expressed as a polynomial constraint on the coefficients of $f_{ijk}(\mathbf{v})$, and hence as a polynomial constraint $c(\mathbf{Q}_i, \mathbf{Q}_j, \mathbf{Q}_k) = 0$ on $\mathbf{Q}_i, \mathbf{Q}_j, \mathbf{Q}_k$. Either $c(\mathbf{Q}_i, \mathbf{Q}_j, \mathbf{Q}_k) = 0$ for all $\mathbf{Q}_i, \mathbf{Q}_j, \mathbf{Q}_k$ or $c(\mathbf{Q}_i, \mathbf{Q}_j, \mathbf{Q}_k) \neq 0$ in general, thus it suffices to find just one triple $\mathbf{Q}_i, \mathbf{Q}_j, \mathbf{Q}_k$ for which $c(\mathbf{Q}_i, \mathbf{Q}_j, \mathbf{Q}_k) \neq 0$, or equivalently for which $f_{ijk}(\mathbf{v})$ does not possess a singular point. To this end, select

$$\begin{aligned} \mathbf{Q}_i &= (-1/2\sqrt{2}, \sqrt{3}/2\sqrt{2}, 1/\sqrt{2}) \\ \mathbf{Q}_j &= (-1/2\sqrt{2}, \sqrt{3}/2\sqrt{2}, 1/\sqrt{2}) \\ \mathbf{Q}_k &= (0, 0, 1) \end{aligned}$$

On substituting the above values of $\mathbf{Q}_i$, $\mathbf{Q}_j$, $\mathbf{Q}_k$ into (5), we obtain

$$f_{ijk}(\mathbf{v}) = \frac{\sqrt{3}}{4}[v_3^2 v_1 + v_3 v_2^2 + \frac{1}{4}(3v_2^2 v_1 - v_1^3)]$$

It can be shown that $\nabla f_{ijk}(\mathbf{v}) \neq 0$ for all $\mathbf{v}$, thus $f_{ijk}(\mathbf{v})$ does not possess a singular point for the three specified values of $\mathbf{Q}_i$, $\mathbf{Q}_j$, $\mathbf{Q}_k$, thus $f_{ijk}(\mathbf{v})$ does not possess a singular point in general.

Two quartic polynomials

The condition that the first four equations of (2) be compatible with a single value of $\boldsymbol{\Omega}$ yields the quartic polynomial constraint $q_1(\mathbf{v}) = 0$ where [3, 6].

$$q_1(\mathbf{v}) \equiv \det \begin{pmatrix} \mathbf{R}_1.\mathbf{v} & (\mathbf{v}.\mathbf{Q}_1)\mathbf{Q}_1 - \mathbf{v} \\ \mathbf{R}_2.\mathbf{v} & (\mathbf{v}.\mathbf{Q}_2)\mathbf{Q}_2 - \mathbf{v} \\ \mathbf{R}_3.\mathbf{v} & (\mathbf{v}.\mathbf{Q}_3)\mathbf{Q}_3 - \mathbf{v} \\ \mathbf{R}_4.\mathbf{v} & (\mathbf{v}.\mathbf{Q}_4)\mathbf{Q}_4 - \mathbf{v} \end{pmatrix}$$

In addition to $q_1(\mathbf{v})$ we require a second quartic constraint arising from (2), namely $q_2(\mathbf{v}) = 0$ where

$$q_2(\mathbf{v}) \equiv \det \begin{pmatrix} \mathbf{R}_1.\mathbf{v} & (\mathbf{v}.\mathbf{Q}_1)\mathbf{Q}_1 - \mathbf{v} \\ \mathbf{R}_2.\mathbf{v} & (\mathbf{v}.\mathbf{Q}_2)\mathbf{Q}_2 - \mathbf{v} \\ \mathbf{R}_3.\mathbf{v} & (\mathbf{v}.\mathbf{Q}_3)\mathbf{Q}_3 - \mathbf{v} \\ \mathbf{R}_5.\mathbf{v} & (\mathbf{v}.\mathbf{Q}_5)\mathbf{Q}_5 - \mathbf{v} \end{pmatrix}$$

The quartics $q_1(\mathbf{v})$, $q_2(\mathbf{v})$ are linear combinations of the $f_{ijk}(\mathbf{v})$ defined by (5).

It follows from (1) and (2) that any translational velocity compatible with (1) is included amongst the 16 common zeros of $q_1(\mathbf{v})$, $q_2(\mathbf{v})$.

DISTINCT COMMON ZEROS

We prove that $q_1(\mathbf{v})$, $q_2(\mathbf{v})$ have, in general, exactly 16 *distinct* common zeros by obtaining just one pair of quartics $q_1(\mathbf{v})$, $q_2(\mathbf{v})$ with this property. This suffices, because the condition that $q_1(\mathbf{v})$, $q_2(\mathbf{v})$ have one or more multiple common zeros reduces to an algebraic constraint on the coefficients of $\mathbf{Q}_i$, $\dot{\mathbf{Q}}_i$. It requires just one example to show that this algebraic constraint is non-trivial.

Our method is to perturb a given pair $q_1(\mathbf{v})$, $q_2(\mathbf{v})$ by a small amount such that the resulting quartics have only transverse common zeros. This is done in stages as follows. Let $q_1(\mathbf{v})$, $q_2(\mathbf{v})$ have a multiple common zero at $\mathbf{u}$ and n additional transverse common zeros. We show that there exist small perturbations δq_1, δq_2 such that $(q_1 + \delta q_1)(\mathbf{v})$, $(q_2 + \delta q_2)(\mathbf{v})$ have a transverse common zero at $\mathbf{u}$. If δq_1, δq_2 are sufficiently small then the n transverse common zeros of $q_1(\mathbf{v})$, $q_2(\mathbf{v})$ are preserved, because transverse common zeros are stable. It follows that $(q_1 + \delta q_1)(\mathbf{v})$, $(q_2 + \delta q_2)(\mathbf{v})$ have at least $n + 1$ transverse common zeros. On repeating this process at most 15 times we obtain the required pair of quartics.

The proof that suitable δq_1, δq_2 can always be found proceeds as follows. Let $q_1(\mathbf{v})$, $q_2(\mathbf{v})$ have a multiple common zero at $\mathbf{v} = \mathbf{u}$. Then

Case 1: $f_{123}(\mathbf{u}) \neq 0$. We subject $\mathbf{R}_4$, $\mathbf{R}_5$ of (2) to perturbations $\delta\mathbf{R}_4$, $\delta\mathbf{R}_5$ such that $\delta\mathbf{R}_4.\mathbf{u} = \delta\mathbf{R}_5.\mathbf{u} = 0$. Let $\delta q_1(\mathbf{v})$ and $\delta q_2(\mathbf{v})$ be the corresponding perturbations of $q_1(\mathbf{v})$ and $q_2(\mathbf{v})$. We have $\delta q_1(\mathbf{u}) = \delta q_2(\mathbf{u}) = 0$ and

$$\nabla \delta q_1(\mathbf{u}) = -f_{123}(\mathbf{u})\delta\mathbf{R}_4$$

$$\nabla \delta q_2(\mathbf{u}) = -f_{123}(\mathbf{u})\delta\mathbf{R}_5$$

It is thus possible to choose $\delta\mathbf{R}_4$ and $\delta\mathbf{R}_5$ such that $\nabla(q_1 + \delta q_1)(\mathbf{u})$ and $\nabla(q_2 + \delta q_2)(\mathbf{u})$ are non-zero and non-parallel.

Case 2: $f_{123}(\mathbf{u}) = 0$, $f_{124}(\mathbf{u}) \neq 0$. We subject $\mathbf{R}_3$, $\mathbf{R}_5$ to perturbations $\delta\mathbf{R}_3$, $\delta\mathbf{R}_5$ such that $\delta\mathbf{R}_3.\mathbf{u} = 0$, $\delta\mathbf{R}_5.\mathbf{u} \neq 0$. We obtain

$$\nabla \delta q_1(\mathbf{u}) = f_{124}(\mathbf{u})\delta\mathbf{R}_3$$

$$\nabla \delta q_2(\mathbf{u}) = f_{125}(\mathbf{u})\delta\mathbf{R}_3 - (\delta\mathbf{R}_5.\mathbf{u})\nabla f_{123}(\mathbf{u})$$

We have seen that the $f_{ijk}(\mathbf{v})$ do not, in general, possess a singular point, thus we assume $\nabla f_{123}(\mathbf{u}) \neq 0$. It follows that $\delta\mathbf{R}_3$, $\delta\mathbf{R}_5$ can be found such that $\nabla(q_1 + \delta q_1)(\mathbf{u})$, $\nabla(q_2 + \delta q_2)(\mathbf{u})$ are non-zero and non-parallel.

A series of arguments similar to that given in case 2 shows that suitable perturbations δq_1, δq_2 can be found provided at least one of $f_{123}(\mathbf{u})$, $f_{ij4}(\mathbf{u})$, $f_{ij5}(\mathbf{u})$ is non-zero, where $1 \leq i < j \leq 3$. This is sufficient to show that suitable δq_1, δq_2 can be found, because if all such $f_{ijk}(\mathbf{u})$ are equal to zero then the vectors $(\mathbf{u}.\mathbf{Q}_i)\mathbf{Q}_i - \mathbf{u}$ would be coplanar, contradicting result (ii) above.

ZEROS AND THE 3D MOTION

We have seen that any translational velocity $\mathbf{v}$ compatible with (1) is a common zero of $q_1(\mathbf{v})$, $q_2(\mathbf{v})$. The converse result does not hold however because $q_1(\mathbf{v})$, $q_2(\mathbf{v})$ have common zeros *not* compatible with (1). We show that a translational velocity $\mathbf{u}$ is compatible with (1) if and only if $\mathbf{u}$ is a common zero of $q_1(\mathbf{v})$ and $q_2(\mathbf{v})$ and if, in addition, the vectors

$$(\mathbf{R}_i.\mathbf{v}, (\mathbf{v}.\mathbf{Q}_i)\mathbf{Q}_i - \mathbf{v}) \quad i = 1, 2, 3 \tag{6}$$

are linearly independent at $\mathbf{v} = \mathbf{u}$.

Suppose firstly that $\mathbf{u}$ is a common zero of $q_1(\mathbf{v})$, $q_2(\mathbf{v})$ such that the vectors of (6) are linearly *independent* at $\mathbf{v} = \mathbf{u}$. By hypothesis $q_1(\mathbf{u}) = q_2(\mathbf{u}) = 0$, thus there exist non-zero vectors $\mathbf{W}_1, \mathbf{W}_2$ such that [6]

$$\begin{pmatrix} \mathbf{R}_1.\mathbf{u} & (\mathbf{u}.\mathbf{Q}_1)\mathbf{Q}_1 - \mathbf{u} \\ \mathbf{R}_2.\mathbf{u} & (\mathbf{u}.\mathbf{Q}_2)\mathbf{Q}_2 - \mathbf{u} \\ \mathbf{R}_3.\mathbf{u} & (\mathbf{u}.\mathbf{Q}_3)\mathbf{Q}_3 - \mathbf{u} \\ \mathbf{R}_4.\mathbf{u} & (\mathbf{u}.\mathbf{Q}_4)\mathbf{Q}_4 - \mathbf{u} \end{pmatrix} \mathbf{W}_1 = 0 \tag{7}$$

and

$$\begin{pmatrix} \mathbf{R}_1.\mathbf{u} & (\mathbf{u}.\mathbf{Q}_1)\mathbf{Q}_1 - \mathbf{u} \\ \mathbf{R}_2.\mathbf{u} & (\mathbf{u}.\mathbf{Q}_2)\mathbf{Q}_2 - \mathbf{u} \\ \mathbf{R}_3.\mathbf{u} & (\mathbf{u}.\mathbf{Q}_3)\mathbf{Q}_3 - \mathbf{u} \\ \mathbf{R}_5.\mathbf{u} & (\mathbf{u}.\mathbf{Q}_5)\mathbf{Q}_5 - \mathbf{u} \end{pmatrix} \mathbf{W}_2 = 0 \tag{8}$$

The vectors $\mathbf{W}_1, \mathbf{W}_2$ are both normal to the subspace of $\mathcal{R}^4$ spanned by the vectors of (6). By hypothesis, this subspace is of dimension three in $\mathcal{R}^4$, thus $\mathbf{W}_1$ and $\mathbf{W}_2$ are parallel. We scale $\mathbf{W}_1$ and $\mathbf{W}_2$ such that $\mathbf{W}_1 = \mathbf{W}_2$.

We denote $(\mathbf{W}_i)_j$ by W_{ij}, and we define $\mathbf{w} \equiv (w_1, w_2, w_3)$ by $w_j = -W_{1\ j+1}/W_{11}$. (We can assume $W_{11} \neq 0$ since in the case $W_{11} = 0$ we obtain from (7), (8) and the hypothesis that the vectors of (6) are linearly independent the result $\mathbf{W}_1 = \mathbf{W}_2 = 0$, contradicting the definitions of $\mathbf{W}_1$, $\mathbf{W}_2$.) Equations (7), (8), and the definition of $\mathbf{w}$ yield

$$\mathbf{R}_i.\mathbf{u} = (\mathbf{u}.\mathbf{Q}_i)(\mathbf{w}.\mathbf{Q}_i) - \mathbf{u}.\mathbf{w} \tag{9}$$

It follows from (9) that $\mathbf{v} = \mathbf{u}$, $\mathbf{\Omega} = \mathbf{w}$ is a rigid velocity compatible with (1).

Next, suppose that $\mathbf{u}$ is a common zero of $q_1(\mathbf{v})$, $q_2(\mathbf{v})$ such that the three vectors of (6) are linearly *dependent*. We show that $\mathbf{u}$ is in general *not* a translational velocity compatible with the 2D motion field.

It follows from our hypothesis concerning $\mathbf{u}$ that the vectors $(\mathbf{u}.\mathbf{Q}_i)\mathbf{Q}_i - \mathbf{u}$, $i = 1, 2, 3$ are contained in a single plane Π. The vectors $(\mathbf{u}.\mathbf{Q}_i)\mathbf{Q}_i - \mathbf{u}$ $(1 \leq i \leq 5)$ span $\mathcal{R}^3$, thus at least one of $(\mathbf{u}.\mathbf{Q}_4)\mathbf{Q}_4 - \mathbf{u}$, $(\mathbf{u}.\mathbf{Q}_5)\mathbf{Q}_5 - \mathbf{u}$ is not contained in Π. It follows that $\mathbf{R}_4$, $\mathbf{R}_5$ and $\mathbf{Q}_4$, $\mathbf{Q}_5$ can be chosen such that

$$\det \begin{pmatrix} \mathbf{R}_1.\mathbf{u} & (\mathbf{u}.\mathbf{Q}_1)\mathbf{Q}_1 - \mathbf{u} \\ \mathbf{R}_2.\mathbf{u} & (\mathbf{u}.\mathbf{Q}_2)\mathbf{Q}_2 - \mathbf{u} \\ \mathbf{R}_4.\mathbf{u} & (\mathbf{u}.\mathbf{Q}_4)\mathbf{Q}_4 - \mathbf{u} \\ \mathbf{R}_5.\mathbf{u} & (\mathbf{u}.\mathbf{Q}_5)\mathbf{Q}_5 - \mathbf{u} \end{pmatrix} \neq 0$$

The choice of $\mathbf{R}_4$, $\mathbf{R}_5$ and $\mathbf{Q}_4$, $\mathbf{Q}_5$ does not affect the condition $q_1(\mathbf{u}) = q_2(\mathbf{u}) = 0$, since we are assuming that the vectors of (6) are linearly dependent. It follows that $\mathbf{u}$ is in general not compatible with (2).

THE MAIN RESULT

We have seen that a possible translational velocity $\mathbf{v}$ is compatible with the 2D motion field if and only if $\mathbf{v}$ is a common zero of $q_1(\mathbf{v})$, $q_2(\mathbf{v})$ such that the vectors of (6) are linearly independent, and we have shown that $q_1(\mathbf{v})$, $q_2(\mathbf{v})$ have 16 distinct common zeros. We now show that there are exactly six values of $\mathbf{v}$ such that the vectors of (6) are linearly dependent. The main result that there are, in general, exactly $10 = 16$-6 values of $\mathbf{v}$ compatible with a 2D motion field containing just five flow vectors then follows.

Define the matrix A by

$$A = \begin{pmatrix} \mathbf{R}_1.\mathbf{v} & (\mathbf{v}.\mathbf{Q}_1)\mathbf{Q}_1 - \mathbf{v} \\ \mathbf{R}_2.\mathbf{v} & (\mathbf{v}.\mathbf{Q}_2)\mathbf{Q}_2 - \mathbf{v} \\ \mathbf{R}_3.\mathbf{v} & (\mathbf{v}.\mathbf{Q}_3)\mathbf{Q}_3 - \mathbf{v} \end{pmatrix} \equiv (A_1\mathbf{v}|A_2\mathbf{v}|A_3\mathbf{v}|A_4\mathbf{v})$$

where the A_i are 3×3 matrices with coefficients independent of $\mathbf{v}$. Let $g(\mathbf{v}) \equiv \det(A_1\mathbf{v}|A_2\mathbf{v}|A_3\mathbf{v})$. We recall that the row rank of a matrix is equal to the column rank of a matrix. The rows of A are linearly dependent if and only if $\mathbf{v}$ satisfies both

$$g(\mathbf{v}) = f_{123}(\mathbf{v}) = 0 \tag{10}$$

and

$$A_2\mathbf{v} \times A_3\mathbf{v} \neq 0 \tag{11}$$

There are at most nine distinct values of $\mathbf{v}$ satisfying (10) because $g(\mathbf{v})$, $f_{123}(\mathbf{v})$ are cubic plane curves, and amongst these there are at most three distinct values of $\mathbf{v}$, corresponding to the roots of the eigenvalue equation

$$\det(A_2 - \lambda A_3) = 0$$

for which (11) fails to hold.

Direct calculation reveals that there are in general exactly three distinct values of $\mathbf{v}$ for which $A_2\mathbf{v} \times A_3\mathbf{v} = 0$. (To simplify matters, coordinates can be chosen such that $\mathbf{Q}_1 = (1, 0, 0)$ and $(\mathbf{Q}_2)_3 = 0$.) It is thus sufficient to show that there are, in general, exactly nine values of $\mathbf{v}$ satisfying (10), and for this it is sufficient to produce a *single* example for which there are nine distinct values of $\mathbf{v}$ satisfying (10). The example is constructed as follows.

Set $\mathbf{R}_1 = \mathbf{R}_2 = 0$. Then the first two rows of A are linearly dependent if and only if $\mathbf{v} = \mathbf{Q}_1$, $\mathbf{Q}_2$ or $\mathbf{Q}_1 \times \mathbf{Q}_2$ (as points of $\mathcal{P}^2$). Suppose that $\mathbf{v}$ is not equal to $\mathbf{Q}_1$, $\mathbf{Q}_2$ or $\mathbf{Q}_1 \times \mathbf{Q}_2$. In this case (with $\mathbf{R}_1 = \mathbf{R}_2 = 0$) the rows of A are linearly dependent if and only if

$$\mathbf{R}_3.\mathbf{v} = 0 \qquad \text{and} \qquad f_{123}(\mathbf{v}) = 0 \tag{12}$$

There are, in general, exactly three values of $\mathbf{v}$ satisfying (12), because $\mathbf{R}_3.\mathbf{v} = 0$ is a linear constraint on $\mathbf{v}$ and $f_{123}(\mathbf{v}) = 0$ is a cubic constraint on $\mathbf{v}$. In general $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_1 \times \mathbf{Q}_2$ are not on the line $\mathbf{R}_3.\mathbf{v} = 0$, thus we obtain a total of six distinct values of $\mathbf{v}$ for which the rows of A are linearly dependent, and these six values of $\mathbf{v}$ satisfy (10).

It follows that if $\mathbf{R}_1 = \mathbf{R}_2 = 0$ then there are exactly $9 = 6 + 3$ distinct values of $\mathbf{v}$ satisfying (10), thus there are, in general, exactly nine distinct values of $\mathbf{v}$ satisfying (10), hence there are, in general, exactly six values of $\mathbf{v}$ for which the rows of A are linearly independent.

AN ALTERNATIVE METHOD

The above proof that there are, in general, exactly ten distinct rigid velocities compatible with five given image velocities is closely tied to the image plane. A sketch of an alternative proof of the same result is now given based on constructions in 3D space and the theory of ambiguous surfaces [3, 4].

We recall from [4] that if points $\mathbf{P}$ in space moving rigidly with velocity $\mathbf{v}$, $\mathbf{\Omega}$ give rise to image velocities that are also compatible with a second rigid velocity $\mathbf{v}'$, $\mathbf{\Omega}'$ then the points $\mathbf{P}$ lie on the quadric surface

$$\mathbf{l}.\mathbf{P} = (\mathbf{W}'.\mathbf{P})(\mathbf{v}'.\mathbf{P}) - (\mathbf{W}'.\mathbf{v}')(\mathbf{P}.\mathbf{P}) \tag{13}$$

where $\mathbf{l} = \mathbf{v}' \times \mathbf{v}$ and $\mathbf{W}' = \mathbf{\Omega} - \mathbf{\Omega}'$. Equation (13) can be written in the form

$$\mathbf{P}M\mathbf{P} + \mathbf{l}.\mathbf{P} = 0 \tag{14}$$

where M is a symmetric matrix and $\mathbf{l}$ is a vector satisfying $\mathbf{l}.\mathbf{v} = 0$.

Let five image velocities $\dot{\mathbf{Q}}_i$ with base points $\mathbf{Q}_i$ be given, together with a single compatible rigid velocity $\mathbf{v}$, $\mathbf{\Omega}$. Then we can construct points $\mathbf{P}_i$ in space with velocities $\dot{\mathbf{P}}_i = \mathbf{v} + \mathbf{\Omega} \times \mathbf{P}_i$ such that $\mathbf{P}_i$ projects to $\mathbf{Q}_i$ and $\dot{\mathbf{P}}_i$ projects to $\dot{\mathbf{Q}}_i$. Let $\mathcal{S}$ be the space of all quadrics of the form (14) that contain the five points $\mathbf{P}_i$. The quadrics of $\mathcal{S}$ are subject to six linear constraints, namely, $\mathbf{l}.\mathbf{v} = 0$ and

$$\mathbf{P}_i M \mathbf{P}_i + \mathbf{l}.\mathbf{P}_i = 0$$

thus $\mathcal{S}$ has dimension two and hence $\mathcal{S}$ is a copy of $\mathcal{P}^2$ embedded in the space of all quadrics. Each rigid velocity $\mathbf{v}'$, $\mathbf{\Omega}'$ distinct from $\mathbf{v}$, $\mathbf{\Omega}$ but compatible with the $\dot{\mathbf{Q}}_i$, $\mathbf{Q}_i$ yields a quadric in $\mathcal{S}$ with an equation of the form (13). Thus, in order to count the rigid velocities compatible with the $\dot{\mathbf{Q}}_i$, $\mathbf{Q}_i$ it suffices to count the quadrics in $\mathcal{S}$ of the form (13).

Let ψ be the element of $\mathcal{S}$ specified by a pair M, $\mathbf{l}$, and define the matrix N by

$$N = M - \frac{1}{2}\text{Trace}(M)I$$

The quadric ψ has an equation of the form (13) for some choice of $\mathbf{v}'$, $\mathbf{W}'$ if and only if

$$\det(N) = 0 \qquad \text{and} \qquad \mathbf{l}N\mathbf{l} = 0 \qquad (15)$$

The equations of (15) define two cubic curves in $\mathcal{S}$ which intersect at nine points. We thus obtain $10 = 9 + 1$ rigid velocities compatible with the five image velocities $\dot{\mathbf{Q}}_i$.

CONCLUSION

We have shown that there are, in general, exactly ten rigid velocities compatible with a 2D motion field containing five image velocity vectors. Ten is thus a basic measure of the complexity of the problem of obtaining rigid velocities from 2D motion fields. In this context ten is high, indicating that the problem is intrinsically difficult.

Some of the ten rigid velocities may have complex coordinates, in which case they can be discarded on physical grounds. It may also be possible to discard rigid velocities not yielding feasible positions for the points on the rigid body surface giving rise to the 2D motion field, in that some of the points are behind the camera. An example of ten rigid velocities compatible with five image velocity vectors is obtained in [7] using the REDUCE computer algebra system.

An image velocity field can be regarded as the limit of a sequence of image displacement fields, as the size of the displacement becomes small. As a result it appears that many of the properties of image velocities may carry over to the more complicated case of image displacements. For example, a) Demazure [8] shows that there are, in general, exactly ten rigid displacements compatible with five given image displacements; b) the two classes of ambiguous surfaces associated with image displacements and image velocities, respectively, happen to coincide; and c) the ambiguous surfaces arising from image displacements are subject to two cubic constraints analogous to those quoted in (15) above [9].

Questions about the possible connections between the theories of image displacements and image velocities now arise. For example, let $\mathcal{S}_i$ be a sequence of five point image displacement fields such that $\mathcal{S}_i \rightarrow \mathcal{S}$ where $\mathcal{S}$ is an image velocity field. Let $\mathcal{T}_i$ be the set of ten rigid displacements compatible with $\mathcal{S}_i$ and let $\mathcal{T}$ be the set of ten rigid displacements compatible with $\mathcal{S}$. Then is it the case that $\mathcal{T}_i \rightarrow \mathcal{T}$?

It is known that there exist five point image displacement fields compatible with ten real rigid displacements, of which three are feasible [10], however Horn [11] has carried out experiments indicating that many five point image displacement fields are compatible with exactly four real rigid displacements, only one of which is feasible. It may be possible to obtain some understanding of these results by studying the simpler case of image velocities.

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